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Uniqueness and Existence of Solutions of a Boundary-value Problem for a System of Differential Equations

A. Lasota and Z. Opial [2] have proved that, under certain qualitative assumptions, for the second order differential equation

$$x'' = f(t, x, x')$$

the uniqueness of solutions of the boundary-value problem

$$x(a) = r_1, \quad x(b) = r_2$$

implies their existence.

This note gives a generalization of their result to systems of differential equations on the Euclidean plane R^2,

(1) $$x' = f(t, x) \quad \big(x = (x^1, x^2)\big), \quad f(t, x) = \big(f^1(t, x), f^2(t, x)\big)$$

and non-linear boundary-value conditions

(2) $$g\big(x(a)\big) = r_1, \quad g\big(x(b)\big) = r_2, \quad a \neq b, \quad a, b \in D,$$

where $g: R^2 \to R$, $g(R^2) = R$ and D is an open interval.

It will be assumed in the sequel that:

(C) $f(t, x)$ is continuous in $D \times R^2$ and satisfies any condition assuring the uniqueness of solution of Cauchy problem for (1) and the existence of solutions of (1) in the whole D.

Theorem 1. *Assume that g is of class C^1 on R^2 and that it satisfies the following conditions*:

(A) *For arbitrary $r \in R$, the level curve $K(r) = \{x : g(x) = r\}$ is a simple arc separating R^2.*

(B) *For any bounded intervals $D_0 \subset D$, $R_0 \subset R$ and arbitrary $M > 0$, the set*

$$\{x : |\operatorname{grad} g(x) \cdot f(t, x)| \leqslant M, \ t \in D_0, \ g(x) \in R_0\}$$

is bounded (the dot denotes scalar multiplication).

If for every pair $(a, r_1), (b, r_2) \in D \times R$, $a \neq b$, the boundary-value problem (1), (2) *has at most one solution, then for each such a pair* (1), (2) *has exactly one solution.*

Notice that the result of Lasota and Opial corresponds to the case $g(x^1, x^2) = x^1$, $f^1(t, x^1, x^2) = x^2$. It is obvious that in this case (A) and (B) are automatically satisfied.

Proof. Denote by $x = x(t; t_0, x_0)$ the (unique) solution of (1) satisfying the initial condition $x(t_0; t_0, x_0) = x_0$. For arbitrary chosen, fixed numbers a, b, r_1, define the mapping $h: K(r_1) \to R$ by

$$h(x_0) = g\big(x(b; a, x_0)\big).$$

Obviously h is continuous. Moreover, by the uniqueness of solutions of (1), (2), h is injective. For the same reason, for $\bar{x}, \bar{\bar{x}} \in K(r_1)$,

$$h(\bar{x}) < h(\bar{\bar{x}}) \Rightarrow g\big(x(t; a, \bar{x})\big) < g\big(x(t; a, \bar{\bar{x}})\big) \tag{3}$$

for all $t \in (a, \infty) \cap D$. Hence, from (A) it follows that $h\big(K(r_1)\big)$ is an open interval. To prove the theorem it is sufficient to show that

$$h\big(K(r_1)\big) = R. \tag{4}$$

Suppose that (4) does not hold and let $\sup h\big(K(r_1)\big) = r_0 < \infty$. Then there is a sequence $\{x_n\} \subset K(r_1)$ such that

$$h(x_1) < h(x_2) < \dots, \qquad \lim_{n\to\infty} h(x_n) = r_0. \tag{5}$$

Assume, in addition, that $\{x(b; a, x_n)\}$ has no cluster points. Then from (B) it follows that

$$\operatorname{grad} g\big(x(b; a, x_n)\big) \cdot f\big(b, x(b; a, x_n)\big) \neq 0$$

for sufficiently large n.

Passing, if necessary, to a subsequence, without loss of generality we may assume that

$$\operatorname{grad} g\big(x(b; a, x_n)\big) \cdot f\big(b, x(b; a, x_n)\big) > 0 \quad (n = 1, 2, \dots). \tag{6}$$

From the formula

$$\frac{d}{dt} g\big(x(t; a, x_n)\big) = \operatorname{grad} g\big(x(t; a, x_n)\big) \cdot f\big(t, x(t; a, x_n)\big)$$

and (6) it follows that, for a certain $c \in (a, b)$, $h(x_1) > g\big(x(c; a, x_1)\big) = r_3$.

Hence, by (3) and (5), $g\big(x(c; a, x_n) \in (r_3, r_0)\big)$ $(n = 1, 2, \dots)$.

Put $M_0 = (r_0 - r_3)/(b - c)$ and define the set

$$G = \{x : |\operatorname{grad} g(x) \cdot f(t, x)| \leqslant M_0,\ t \in [c, b],\ g(x) \in [r_3, r_0]\}.$$

There is a sequence $\{t_n\} \subset [c, b]$ such that $x(t_n; a, x_n) \in G$ $(n = 1, 2, \dots)$. In fact, $x(t; a, x_n) \in G$ for $t \in [c, b]$ implies that $\dfrac{d}{dt} g\big(x(t; a, x_n)\big) > M_0$ in $[c, b]$. Integrating this inequality over $[c, b]$ gives

$$g\big(x(b : a, x_n)\big) - g\big(x(c : a, x_n)\big) > r_0 - r_3,$$

but this contradicts $g\big(x(c; a, x_n)\big) \in (r_3, r_0)$.

Since G is bounded, $\{x(t_n; a, x_n)\}$ has a cluster point. Passing to appropriate subsequences, we may assume that

(7) $$t_n \to t_0, \quad x(t_n; a, x_n) = z_n \to z_0 \quad \text{as} \quad n \to \infty.$$

Since $g(x(a; t_n, z_n)) = g(x(a; a, x_n)) = r_1$, $g(x(b; t_n, z_n)) = g(x(b; a, x_n)) = h(x_n)$, (7), (3) and the continuous dependence of solutions on their initial values imply that

(8) $$g(x(b; t_0, z_0)) = r_0, \quad g(x(a; t_0, z_0)) = r_1,$$

which is impossible since $h(K(r_1))$ is open.

If $\{x(b; a, x_n)\}$ has a convergent subsequence $\{x(b; a, x_{k(n)})\}$, then taking $t_{k(n)} = b$, $z_{k(n)} = x(t_{k(n)}; a, x_{k(n)})$ we obtain again (8), which completes the proof of Theorem 1.

Remark. From the above proof it is clear that (2) may be replaced by the more general boundary-value conditions

(9) $$g^1(x(a)) = r_1, \quad g^2(x(b)) = r_2,$$

where $g^2(x)$ satisfies (A) and (B), $g^1(x)$ satisfies (A) and the condition:

For arbitrary r_1, r_2 the intersection of sets $\{x : g^1(x) = r_1\}$, $\{x : g^2(x) = r_2\}$ is either empty or a one-point set.

This last condition is necessary for the uniqueness of boundary-value problem (1), (9).

In the case of linear boundary conditions

(10) $$x^1(a) = r_1, \quad x^1(b) = r_2$$

one can obtain as a consequence of Theorem 1 the following

Theorem 2. *Let* $f(t, x)$ *satisfy* (C), *the condition*

(11) $$\lim_{|x^2| \to \infty} |f^1(t, x^1, x^2)| = \infty$$

uniformly with respect to (t, x^1) *in every compact subset of* $D \times R$; and

(D) $f^i(t, x^1, x^2)$ *is increasing with respect to* $x_j (j \neq i)$, $i = 1, 2$.

Then boundary-value problem (1), (10) *has exactly one solution for every pair* $(a, r_1), (b, r_2) \in D \times R$.

Proof. It is easy to verify that, for $g(x) = x^1$, (B) and (11) are equivalent. Condition (D) implies the uniqueness of the boundary-value problem (1), (10) (see [1], p. 28). Hence, by Theorem 1, the assertion of Theorem 2 holds true.

REFERENCES

[1] P. Hartman, *Ordinary differential equations*, N. York 1964.

[2] A. Lasota, Z. Opial, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second-order differential equation*, Colloquium Math. 18 (1967), 1—5.