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On a Nonlinear Integral Inequality

1. It is well known that if a real-valued, nonnegative and measurable function $x\colon [0, T)\to R$, $0 < T \leqslant +\infty$, satisfies the linear integral inequality

$$(1)\qquad x(t) \leqslant f(t)+p(t)\int_0^t q(s)x(s)ds \quad \text{for} \quad t\in[0, T),$$

where the functions f, p, q are nonnegative and measurable in $[0, T)$, then

$$(2)\qquad x(t) \leqslant f(t)+p(t)\int_0^t q(s)f(s)\exp\Big(\int_0^t p(r)q(r)dr\Big)ds \quad \text{for} \quad t\in[0, T).$$

Inequality (2) follows immediately from the fact that its right-hand side is the solution of the linear integral equation

$$(3)\qquad y(t) = f(t)+p(t)\int_0^t q(s)y(s)ds,$$

and that every function x satisfying inequality (1) is smaller or equal to the solution of (3).

Similarly, the problem of finding an estimate for a function x satisfying the inequality

$$(4)\qquad x(t) \leqslant f(t)+p(t)\int_0^t q(s)x^n(s)ds \quad \text{for} \quad t\in[0, T)$$

can be reduced to the problem of finding the solution of the nonlinear integral equation

$$(5)\qquad y(t) = f(t)+p(t)\int_0^t q(s)y^n(s)ds,$$

since from the general comparative theorem of Satō [2] it follows that if a function x satisfies inequality (4), then $x(t) \leqslant y(t)$, where y is the solution of equation (5). Since, however, we are not able to solve equation (5) in a closed form, this solution to our problem is only theoretical and therefore any approximate estimate for a function x satisfying (4) may be of practical importance and the search for such estimates remains still an open question. This situation is further complicated by the fact that the solution of equation (5) exists, in general, only on a certain subinterval of the interval $[0, T)$. Thus, various estimates should be compared from the point of view of the precision and of the length of intervals in which they are valid.

One of such estimates has been given in a recent paper [1] by Maroni who has proved the following

Theorem. *Let f, p, q be nonnegative and measurable in $[0, T)$. Then each nonnegative and measurable function x satisfying inequality (4) satisfies, for $n = 2$, the inequality*

$$(6)\qquad x(t) \leqslant \frac{H_2(t)}{1-\int\limits_0^t p(s)q(s)H_2(s)\exp\left(-\int\limits_s^t p(r)q(r)f(r)\,dr\right)ds}$$

and, for $n \geqslant 3$, the inequality

$$(7)\qquad x(t) \leqslant \frac{H_n(t)}{\left\{1-(n-1)(n-2)\int\limits_0^t p(s)q(s)H_n(s)\{H_n^{n-2}(s)-f(s)^{n-2}\}\exp\left(-\frac{(n-1)(n-2)}{2}\int\limits_s^t p(r)q(r)H_n(r)f(r)^{n-2}\,dr\right)ds\right\}^{\frac{1}{2(n-2)}}}$$

for $t \in [0, t_n)$, where

$$H_n(t) = f(t)+p(t)\int\limits_0^t q(s)f^n(s)\exp\left(\int\limits_s^t p(r)q(r)f^{n-1}(r)\,dr\right)ds$$

and t_n is the first point for which the denominator of the right-hand side of (6) or (7), respectively, is equal to 0.

In the present paper, combining the method of Maroni with the idea of changing integral inequalities into differential ones introduced by T. Ważewski [4], we give other estimates for a function x satisfying inequality (4). In addition we shall show that in some cases the estimates presented here are better than those of Maroni and are satisfied in longer intervals.

2. We assume throughout this paper that functions f, p, q and x are nonnegative and measurable in $[0, T)$ and that all considered integrals do exist.

Theorem 1. *If the function f/p is nondecreasing and measurable in $[0, T)$, then each function x satisfying inequality (4) satisfies the inequality*

$$(8)\qquad x(t) \leqslant \frac{f(t)}{\left(1-(n-1)\int\limits_0^t q(s)p(s)f^{n-1}(s)\,ds\right)^{\frac{1}{n-1}}}$$

for $t \in [0, t_n)$, *where* t_n *is the first point for which the denominator of the right-hand side of* (8) *is equal to* 0.

The proof of this Theorem is based on two lemmas.

Lemma 1. *If the function z satisfies the inequality*

$$z'(t) \leqslant p(t)z(t)+f(t) \quad \text{for} \quad t \in [0, T),$$

then

$$z(t) \leqslant \exp\Big(\int_0^t p(s)\,ds\Big)\Big[z(0)+\int_0^t f(s)\exp\Big(-\int_0^s p(r)\,dr\Big)ds\Big] \tag{9}$$

for $t \in [0, T)$.

Proof. Since the right-hand side of (9) is the solution of the linear differential equation

$$y'(t) = p(t)y(t)+f(t)$$

satisfying the initial value condition $y(0) = z(0)$, Lemma 1 follows from the general theory of differential inequalities (see [3]).

Lemma 2. *Let the function f/p satisfy the assumptions of Theorem* 1 *and let x satisfy the inequality*

$$x(t) \leqslant f(t)+p(t)\int_0^t q(s)x^2(s)\,ds \quad \text{for} \quad t \in [0, T). \tag{10}$$

Then

$$x(t) \leqslant \frac{f(t)}{1-\int_0^t q(s)p(s)f(s)\,ds} \quad \text{for} \quad t \in [0, t_2), \tag{11}$$

where t_2 *is the first point for which the denominator of the right-hand side of* (11) *is equal to* 0.

Proof. For the function

$$z(t) = \int_0^t q(s)x^2(s)\,ds$$

from inequality (10) we obtain

$$z'(t) \leqslant q(t)\big(f(t)+p(t)z(t)\big)^2.$$

Setting

$$v(t) = z(t)+\frac{f(t)}{p(t)}$$

we get

$$v'(t) \leqslant q(t)p^2(t)v^2(t)+\Big(\frac{f(t)}{p(t)}\Big)'.$$

From Lemma 1 it follows that

$$v(t) \leqslant \exp\Big(\int^t q(r)p^2(r)v(r)\,dr\Big)\Big(\frac{f(0)}{p(0)}+\int_0^t\Big[\frac{f(s)}{p(s)}\Big]'\exp\Big(-\int_0^s q(r)p^2(r)v(r)\,dr\Big)ds\Big).$$

Since the function f/p is nondecreasing, we have

$$v(t) \leqslant \exp\left(\int_0^t q(r)p^2(r)v(r)\,dr\right)\frac{f(t)}{p(t)}.$$

Multiplying the above inequality by

$$\exp\left(-\int_0^t q(r)p^2(r)v(r)\,dr\right)$$

we have

$$v(t)\exp\left(-\int_0^t q(r)p^2(r)v(r)\,dr\right) \leqslant \frac{f(t)}{p(t)}. \tag{12}$$

Consider the function

$$V(t) = \exp\left(-\int_0^t q(r)p^2(r)v(r)\,dr\right).$$

From inequality (12) we obtain

$$V'(t) \geqslant -q(t)p(t)f(t),$$

therefore

$$V(t) \geqslant 1-\int_0^t q(s)p(s)f(s)\,ds. \tag{13}$$

From the definition of the function V and inequalities (12) and (13) we have

$$v(t) \leqslant \frac{f(t)}{p(t)\left(1-\int_0^t q(s)p(s)f(s)\,ds\right)}$$

Since

$$x(t) \leqslant f(t)+p(t)z(t) = p(t)\left(\frac{f(t)}{p(t)}+z(t)\right) = p(t)v(t),$$

we obtain inequality (11).

Proof of Theorem 1. We shall proceed by induction. From Lemma 2 it follows that Theorem 1 is true for $n = 2$. Assume that it is true for n and let the function x satisfy the inequality

$$x(t) \leqslant f(t)+p(t)\int_0^t q(s)x^{n+1}(s)\,ds \quad \text{for} \quad t \in [0, T).$$

From the inductive assumption we obtain

$$x(t) \leqslant \frac{f(t)}{\left\{1-(n-1)\int_0^t p(s)f^{n-1}(s)q(s)x(s)\,ds\right\}^{\frac{1}{n-1}}}. \tag{14}$$

For the function

$$u(t)=\int_0^t p(s)f^{n-1}(s)q(s)x(s)\,ds$$

from inequality (14) we have the inequality

$$u'(t)\leqslant\frac{p(t)q(t)f^n(t)}{\{1-(n-1)u(t)\}^{\frac{1}{n-1}}}$$

which can be written in the following form

$$\left(-[1-(n-1)u(t)]^{\frac{n}{n-1}}\right)'\leqslant np(t)q(t)f^n(t).$$

From the above inequality we have

$$-[1-(n-1)u(t)]^{\frac{n}{n-1}}+1\leqslant n\int_0^t p(s)q(s)f^n(s)\,ds$$

and therefore

$$\left(1-(n-1)u(t)\right)^{\frac{1}{n-1}}\geqslant\left(1-n\int_0^t p(s)q(s)f^n(s)\,ds\right)^{\frac{1}{n}}. \tag{15}$$

Now, from inequalities (14), (15) and the definition of the function u we obtain the inequality

$$x(t)\leqslant\frac{f(t)}{\left(1-n\int_0^t p(s)q(s)f^n(s)\,ds\right)^{\frac{1}{n}}}$$

as was to be shown.

3. In Theorem 1 the function f/p has been assumed to be nondecreasing. It is easy, however, to give to this theorem a form suitable for a wider class of functions f and p.

Theorem 2. *If the function f/p is bounded and measurable in $[0,T)$, then each function x satisfying inequality* (4) *satisfies the inequality*

$$x(t)\leqslant\frac{p(t)g(t)}{\left(1-(n-1)\int_0^t q(s)p^n(s)g^{n-1}(s)\,ds\right)^{\frac{1}{n-1}}} \tag{16}$$

for $t\in[0,t_n)$, where $g(t)=\sup\left\{\frac{f(s)}{p(s)}:s\in[0,t)\right\}$ and t_n is the first point for which the denominator of the right-hand side of (16) *is equal to* 0.

Proof. Since the function x satisfies inequality (4), we have

$$x(t)\leqslant p(t)g(t)+p(t)\int_0^t q(s)x^n(s)\,ds.$$

Thus, a straightforward application of inequality (8) yields (16) and completes the proof.

4. In order to compare the inequalities of Maroni with the inequalities of our Theorem 1, we put $P(t) = q(t)p(t)f(t)$ and $Q(t) = \frac{f(t)}{p(t)}$ which enables us to write inequalities (6) and (11), respectively, in the following forms

$$x(t) \leqslant \frac{f(t)+p(t)\int\limits_0^t P(s)\,Q(s)\exp\Big(\int\limits_s^t P(r)\,dr\Big)ds}{1-\int\limits_0^t P(s)\exp\Big(-\int\limits_s^t P(u)\,du\Big)\Big(1+\frac{1}{Q(s)}\int\limits_0^s P(r)\,Q(r)\exp\Big(\int\limits_r^s P(u)\,du\Big)dr\Big)ds}$$

$$x(t) \leqslant \frac{f(t)}{1-\int\limits_0^t P(r)\,dr}\,.$$

Theorem 3. *Let the function* $Q(t) = \frac{f(t)}{p(t)}$ *satisfy the assumptions of Theorem* 1 *and let* $Q(t) \neq 0$ *for* $t \in [0, T)$. *If the function*

$$F(r) = Q(r)\exp\Big(\int\limits_r^t P(u)\,du\Big)$$

is nonincreasing for $r \in [0, t]$, *then*

$$\text{(17)}\quad \int\limits_0^t P(s)\exp\Big(-\int\limits_s^t P(u)\,du\Big)\Big(1+\frac{1}{Q(s)}\int\limits_0^s P(r)\,Q(r)\exp\Big(\int\limits_r^s P(u)\,du\Big)dr\Big)ds \geqslant \int\limits_0^t P(r)\,dr\,.$$

Proof. Setting

$$w(t) = \int\limits_0^t P(s)\exp\Big(\int\limits_0^s P(u)\,du\Big)\Big(1+\frac{1}{Q(s)}\int\limits_0^s P(r)\,Q(r)\exp\Big(\int\limits_r^s P(u)\,du\Big)dr\Big)ds$$

$$u(t) = \exp\Big(\int\limits_0^t P(u)\,du\Big)\int\limits_0^t P(r)\,dr$$

we obtain

$$\text{(18)}\quad w'(t) = P(t)\exp\Big(\int\limits_0^t P(u)\,du\Big)\times\Big[1+\frac{1}{Q(t)}\int\limits_0^t P(r)\,Q(r)\exp\Big(\int\limits_r^t P(u)\,du\Big)dr\Big]$$

$$\geqslant P(t)\exp\Big(\int\limits_0^t P(u)\,du\Big)\Big(1+\int\limits_0^t P(r)\,dr\Big) = u'(t)\,.$$

Since $w(0) = u(0) = 0$, we have therefore

$$w(t) \geqslant u(t)\,.$$

Hence, multiplying by $\exp\big(-\int\limits_0^t P(r)\,dr\big)$, we obtain inequality (17).