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On a Boundary Value Problems for Systems of Ordinary Differential Equations of Second Order

INTRODUCTION

In the present note we are concerned with a system of ordinary differential equations

$$(0.1) \quad x_i'' = f_i(t, x_1, \dots, x_m, x_1', \dots, x_m') \quad (i = 1, \dots, m),$$

and the boundary value problem

$$(0.2) \quad x_i(0) = x_i(h) = 0 \quad (i = 1, \dots, m).$$

Problem (0.1)—(0.2) can be written shortly in vector notation as

$$(0.3) \quad x'' = f(t, x, x'), \quad x(0) = x(h) = 0,$$

where $x = (x_1, \dots, x_m)$, $f = (f_1, \dots, f_m)$.

By a *solution* of (0.3) we mean any function $x: [0, h] \rightarrow R^m$ with absolutely continuous derivative $x'(t)$, satisfying (0.3) almost everywhere on $[0, h]$.

Problem (0.3) has been considered by I. Kovač and I. Savtchenko [3]; they have given a solution to this problem in the case of $f(t, x, x') = f(t, x)$. In the present note, making use of the method of A. Lasota and Z. Opial [4], we give a generalization and an extension of their results.

Section 1 contains some inequalities of Wirtinger's and Opial's type. Section 2 is devoted to some differential inequalities and in Section 3 we formulate and prove the main theorems.

1. INEQUALITIES

For $x \in R^m$, $|x|$ denotes the Euclidean norm of x . In what follows all functions $x: [0, h] \rightarrow R^m$ are supposed to be absolutely continuous and such that $x(0) = x(h) = 0$.

Inequality 1. If the derivative $x'(t)$ is absolutely continuous, then

$$\int_0^h |x'(t)|^2 dt \leq \int_0^h |x(t)| |x''(t)| dt.$$

The equality occurs for $x(t) = (x_1(t), \dots, x_m(t))$, where $x_i(t) = A \sin \frac{\pi t}{h}$ ($i = 1, \dots, m$).

Inequality 2 ([2], p. 184). Under the general assumptions,

$$\int_0^h |x(t)|^2 dt \leq \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt.$$

The equality occurs for $x(t) = (x_1(t), \dots, x_m(t))$, where $x_i(t) = A \sin \frac{\pi t}{h}$ ($i = 1, \dots, m$).

Inequality 3 [6]. Under the general assumptions,

$$\int_0^h x(t) |x'(t)| dt \leq \frac{h}{4} \int_0^h |x'(t)|^2 dt.$$

The equality occurs for $x(t) = (x_1(t), \dots, x_m(t))$, where

$$x_i(t) = \begin{cases} \frac{1}{2}t & \text{for } 0 \leq t \leq \frac{h}{2} \\ \frac{1}{2}(h-t) & \text{for } \frac{h}{2} \leq t \leq h \end{cases} \quad (i = 1, \dots, m).$$

Inequality 4. For any summable function $p: [0, h] \rightarrow [0, +\infty)$,

$$\int_0^h p(t) |x(t)|^2 dt < \frac{h}{4} \int_0^h p(t) dt \int_0^h |x'(t)|^2 dt.$$

This inequality is the best possible and we have the equality only if $x(t) \equiv 0$ or $p(t) = 0$ almost everywhere in $[0, h]$.

Inequality 5. For any square summable function $p: [0, h] \rightarrow [0, +\infty)$,

$$\int_0^h p(t) |x(t)| |x'(t)| dt \leq \left(\frac{h}{4} \int_0^h p^2(t) dt \right)^{\frac{1}{2}} \int_0^h |x'(t)|^2 dt.$$

Proof of Inequality 1. For $x(t) = (x_1(t), \dots, x_m(t))$, $x(0) = x(h) = 0$, integrating by parts yields

$$\int_0^h x_i(t) x_i''(t) dt = - \int_0^h x_i'^2(t) dt$$

and hence

$$\int_0^h x_i'^2(t) dt \leq \int_0^h |x_i(t)| |x_i''(t)| dt.$$

Using the Cauchy inequality, we obtain

$$\begin{aligned} \int_0^h |x'(t)|^2 dt &= \sum_{i=1}^m \int_0^h x_i'^2(t) dt \leq \sum_{i=1}^m \int_0^h |x_i(t)| |x_i''(t)| dt \\ &\leq \int_0^h \left(\sum_{i=1}^m x_i^2(t) \right)^{\frac{1}{2}} \left(\sum_{i=1}^m x_i''^2(t) \right)^{\frac{1}{2}} dt = \int_0^h |x(t)| |x''(t)| dt. \end{aligned}$$

Proof of Inequality 2. From the well-known Wirtinger inequality [2] it follows that

$$\begin{aligned} \int_0^h |x(t)|^2 dt &= \sum_{i=1}^m \int_0^h x_i^2(t) dt \\ &\leq \sum_{i=1}^m \frac{h^2}{\pi^2} \int_0^h x_i'^2(t) dt = \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt. \end{aligned}$$

Proof of Inequality 3. Our proof is a simple modification (for the vector case) of the proof given in [5] (see also [6]) for the one dimensional version of this inequality.

For

$$y(t) = \int_0^t |x'(t)| dt \quad \text{and} \quad z(t) = \int_t^h |x'(t)| dt$$

we have $y'(t) = |x'(t)| = -z'(t)$ and $|x(t)| \leq y(t)$, $|x(t)| \leq z(t)$ for $t \in [0, h]$.

Hence

$$\begin{aligned} \int_0^{h/2} |x(t)| |x'(t)| dt &\leq \int_0^{h/2} y(t) y'(t) dt = \frac{1}{2} y^2 \left(\frac{h}{2} \right), \\ \int_{h/2}^h |x(t)| |x'(t)| dt &\leq - \int_{h/2}^h z(t) z'(t) dt = \frac{1}{2} z^2 \left(\frac{h}{2} \right). \end{aligned}$$

Thus

$$\int_0^h |x(t)| |x'(t)| dt \leq \frac{1}{2} \left[y^2 \left(\frac{h}{2} \right) + z^2 \left(\frac{h}{2} \right) \right].$$

Furthermore, using the Buniakowski inequality we get

$$y^2 \left(\frac{h}{2} \right) \leq \frac{h}{2} \int_0^{h/2} |x'(t)|^2 dt, \quad z^2 \left(\frac{h}{2} \right) \leq \frac{h}{2} \int_{h/2}^h |x'(t)|^2 dt$$

and this proves Inequality 3.

Proof of Inequality 4. We have, for every $t \in [0, h]$,

$$|x(t)| \leq \int_0^t |x'(t)| dt, \quad |x(t)| \leq \int_t^h |x'(t)| dt.$$

Hence

$$|x(t)| \leq \frac{1}{2} \int_0^h |x'(t)| dt$$

and therefore

$$(1.1) \quad \int_0^h p(t) |x(t)|^2 dt \leq \frac{1}{4} \int_0^h p(t) dt \left[\int_0^h |x'(t)| dt \right]^2$$

Finally, applying the Schwartz inequality

$$(1.2) \quad \left[\int_0^h |x'(t)| dt \right]^2 \leq h \int_0^h |x'(t)|^2 dt$$

we get Inequality 4. It is easy to see that equalities in (1.1) and (1.2) with assumption $\int_0^h p(t) dt > 0$ imply $x(t) \equiv 0$ in $[0, h]$.

The following example shows that, if we replace $\frac{h}{4}$ by any number $\alpha < \frac{h}{4}$, then Inequality 4 fails to be true. Let

$$x_i(t) = \begin{cases} t & \text{for } 0 \leq t \leq k \\ k & \text{for } k \leq t \leq h-k \\ -t+k & \text{for } h-k \leq t \leq h, \end{cases}$$

$$p(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq k \\ 1 & \text{for } k < t < h-k \\ 0 & \text{for } h-k \leq t \leq h, \end{cases}$$

where $2\alpha < k < \frac{h}{2}$, $i = 1, \dots, m$. We have

$$\int_0^h p(t) |x(t)|^2 dt = mk^2(h-2k)$$

$$\int_0^h p(t) dt = h-2k, \quad \int_0^h |x'(t)|^2 dt = 2km.$$

Hence $mk^2(h-2k) \leq 2\alpha mk(h-2k)$, in contradiction with $2\alpha < k$.

Proof of Inequality 5 is an immediate consequence of Inequality 4 and the Schwartz inequality.

$$\int_0^h p(t) |x(t)| |x'(t)| dt \leq \left(\int_0^h p^2(t) |x(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^h |x'(t)|^2 dt \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{h}{4} \int_0^h p^2(t) dt \int_0^h |x'(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_0^h |x'(t)|^2 dt \right)^{\frac{1}{2}} \leq \left(\frac{h}{4} \int_0^h p^2(t) dt \right)^{\frac{1}{2}} \int_0^h |x'(t)|^2 dt.$$

2. DIFFERENTIAL INEQUALITIES

Throughout this Section the function $x : [0, h] \rightarrow R$ is assumed to have absolutely continuous first derivative x' and to vanish at both ends of the interval $[0, h]$, i.e., $x(0) = x(h) = 0$.

Lemma 1. *If*

$$(2.1) \quad |x''(t)| \leq N|x(t)|$$

almost everywhere on $[0, h]$ and

$$(2.2) \quad h < \frac{\pi}{\sqrt{N}},$$

then $x(t) \equiv 0$ in $[0, h]$.

Condition (2.2) is the best possible.

Proof. Inequality 1 and (2.1) imply that

$$\int_0^h |x'(t)|^2 dt \leq \int_0^h |x''(t)||x(t)| dt \leq N \int_0^h |x(t)|^2 dt.$$

Hence, by Inequality 2,

$$\int_0^h |x'(t)|^2 dt \leq N \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt.$$

If $x(t) \not\equiv 0$ in $[0, h]$, then $\int_0^h |x'(t)|^2 dt > 0$ and therefore $1 \leq N \frac{h^2}{\pi^2}$. Thus (2.2)

implies that $x(t) \equiv 0$. For $h = \frac{\pi}{\sqrt{N}}$ the function $x(t) = (x_1(t), \dots, x_m(t))$, where $x_i(t) = A \sin \sqrt{N} t$ ($i = 1, \dots, m$), satisfies (2.1).

Lemma 2. *If*

$$(2.3) \quad |x''(t)| \leq N|x(t)| + M|x'(t)|$$

almost everywhere on $[0, h]$ and

$$(2.4) \quad N \frac{h^2}{\pi^2} + M \frac{h}{4} < 1,$$

then $x(t) \equiv 0$ in $[0, h]$.

Proof. By Inequalities 1, 2, 3 and (2.3) we have

$$\begin{aligned} \int_0^h |x'(t)|^2 dt &\leq \int_0^h |x''(t)||x(t)| dt \leq N \int_0^h |x(t)|^2 dt + M \int_0^h |x(t)||x'(t)| dt \\ &\leq N \frac{h^2}{\pi^2} \int_0^h |x'(t)|^2 dt + M \frac{h}{4} \int_0^h |x'(t)|^2 dt. \end{aligned}$$

Therefore, if $x(t) \neq 0$, we would have $1 \leq N \frac{h^2}{\pi^2} + M \frac{h}{4}$, in contradiction with (2.4).

Lemma 3. If the function $p: [0, h] \rightarrow [0, +\infty)$ is summable, if

$$(2.5) \quad |x''(t)| \leq p(t)|x(t)|$$

almost everywhere on $[0, h]$ and

$$(2.6) \quad \int_0^h p(t) dt \leq \frac{4}{h},$$

then $x(t) \equiv 0$ in $[0, h]$.

Condition (2.6) is the best possible.

Proof. The assumption $x(t) \neq 0$ implies that $\int_0^h |x'(t)|^2 dt > 0$. Since, by Inequalities 1, 4 and (2.5), we have

$$\begin{aligned} \int_0^h |x'(t)|^2 dt &\leq \int_0^h |x''(t)||x(t)| dt \leq \int_0^h p(t)|x(t)|^2 dt \\ &< \frac{h}{4} \int_0^h p(t) dt \int_0^h |x'(t)|^2 dt, \end{aligned}$$

it follows that

$$1 < \frac{h}{4} \int_0^h p(t) dt.$$

A simple modification of the example of Denkowski [1] shows that the number $\frac{4}{h}$ cannot be replaced by any greater.

Lemma 4. If the function $p: [0, h] \rightarrow [0, +\infty)$ is summable and the function $q: [0, h] \rightarrow [0, +\infty)$ is square summable, if the inequality

$$(2.7) \quad |x''(t)| \leq p(t)|x(t)| + q(t)|x'(t)|$$

is fulfilled almost everywhere on $[0, h]$ and

$$(2.8) \quad \frac{h}{4} \int_0^h p(t) dt + \sqrt{\frac{h}{4} \int_0^h q^2(t) dt} < 1,$$

then $x(t) \equiv 0$ in $[0, h]$.

Proof. Using Inequalities 1, 4, 5 and assumption (2.7), we have

$$\int_0^h |x'(t)|^2 dt \leq \int_0^h |x''(t)||x(t)| dt$$

$$\begin{aligned} &\leq \int_0^h p(t)|x(t)|^2 dt + \int_0^h q(t)|x(t)||x'(t)| dt \\ &\leq \frac{h}{4} \int_0^h p(t) dt \int_0^h |x'(t)|^2 dt + \sqrt{\frac{h}{4} \int_0^h q^2(t) dt} \int_0^h |x'(t)|^2 dt. \end{aligned}$$

Hence, for $x(t) \neq 0$, it follows that

$$1 \leq \frac{h}{4} \int_0^h p(t) dt + \sqrt{\frac{h}{4} \int_0^h q^2(t) dt}.$$

in contradiction with (2.8).

3. APPLICATION TO DIFFERENTIAL EQUATIONS

Now we state and prove two theorems which are particular cases of the general theory of Lasota and Opial [4].

We shall say that the function $f: [0, h] \times R^n \rightarrow R^m$ satisfies *condition (C)* (*Carathéodory's condition*), if for every fixed $t \in [0, h]$, $f(t, p)$ is continuous in p and for every fixed $p \in R^n$ $f(t, p)$ is measurable with respect to t .

Theorem 1. *Suppose that in (0, 3) the function $f: [0, h] \times R^m \times R^m \rightarrow R^m$ satisfies condition (C) and the inequality*

$$(3.1) \quad |f(t, x, y)| \leq p(t)|x| + q(t)|y| + g(t, x, y)$$

for $t \in [0, h]$, $x, y \in R^m$, where $p(t)$ and $q(t)$ are summable on $[0, h]$ and the function $g: [0, h] \times R^m \times R^m \rightarrow R$ satisfies condition (C) and the following assumption

$$(3.2) \quad \lim_{k \rightarrow \infty} \frac{1}{k} \int_0^h \sup_{|x|+|y| \leq k} g(t, x, y) dt = 0.$$

Under these assumptions, if $x(t) \equiv 0$ is the unique solution on $[0, h]$ of the inequality

$$(3.3) \quad |x''(t)| \leq p(t)|x(t)| + q(t)|x'(t)|$$

satisfying the boundary condition $x(0) = x(h) = 0$, then problem (0. 3) has at least one solution.

Theorem 2. *Suppose that in (0. 3) the function $f: [0, h] \times R^m \times R^m \rightarrow R^m$ satisfies condition (C) and the Lipschitz inequality*

$$(3.4) \quad |f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq p(t)|u - \bar{u}| + q(t)|v - \bar{v}|$$

for each $u, \bar{u}, v, \bar{v} \in R^m$, where the functions $p(t), q(t)$ are summable on $[0, h]$ and $f(t, 0, 0)$ is summable over $[0, h]$. Under these assumptions, if $x(t) \equiv 0$ is the unique

