ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCLII PRACE MATEMATYCZNE, ZESZYT 15 1971

Andrzej Zajtz

Affine Representations of Groups

Let G be a group and A an n-dimensional affine space over a vector space V. Any homomorphism of G in the group of all affine transformations of the space A will be called an *affine representation* of G.

In a fixed basis $\{0, e_i\}_{1\leqslant i\leqslant n}$ of A to every element $s \in G$ corresponds a pair $(F(s), g(s))$, $\det F(s) \neq 0$, such that the corresponding affine transformation has the form

$$x' = F(s)x+g(s) \quad (x, x' \in A)\,. \tag{1}$$

Here $F(s)$ is a non-singular $n\times n$ matrix and $g(s)$ is an $n\times 1$ matrix. From the group condition it follows that the matrix functions $F(s)$, $g(s)$ define an affine representation of G iff they satisfy two following equations

$$F(st) = F(s)F(t) \quad (s, t \in G) \tag{2}$$

$$g(st) = F(s)g(t)+g(s) \tag{3}$$

with the initial condition $F(1) = E$ (the unit matrix). This condition implies that $F(s)$ is non-singular for any s. The dimension n of A will be called a *degree* of the representation.

To any affine representation $\{F(s), g(s)\}$ of G corresponds an associated linear representation $\{F(s)\}$ which gives the acting

$$v' = F(s)v \quad (v, v' \in V)$$

of G in the vector space V.

Two affine representations R_1 in A_1 and R_2 in A_2 are *equivalent* if there exists an isomorphism $\mu: A_1 \to A_2$ such that

$$R_2(\mu(x)) = \mu(R_1(x)) \tag{4}$$

for all $x \in A_1$. For fixed bases in A_1 and A_2, μ is defined by a non-singular matrix C and a translation vector v, and R_i $(i = 1, 2)$ by the pairs $(F_i(s), g_i(s))$. Then $\mu(x) = Cx+v$ and condition (4) takes the form

$$\begin{aligned} F_1(s) &= C^{-1}F_2(s)C\,,\\ g_1(s) &= C^{-1}\{[F_2(s)-E]v+g_2(s)\}\,. \end{aligned} \tag{5}$$

for all $s \in G$. In particular, representations R_1 and R_2 having the same linear part $F(s)$ are equivalent iff there exists a constant vector v such that

$$(6) \qquad g_1(s)-g_2(s) = \big(F(s)-E\big)v \quad (s \in G).$$

An affine representation $\{F(s),\ 0\}$ can be identified with the linear representation $\{F(s)\}$, so it is affinely trivial. In general, an affine representation $\{F(s), g(s)\}$ will be called *trivial* if it is equivalent to the representation $\{F(s), 0\}$. In view of (6) it is trivial iff there exists a vector v such that

$$(7) \qquad g(s) = \big(F(s)-E\big)v \quad (s \in G).$$

Given a linear representation $\{F(s)\}$ in V, we can form an affine (trivial) representation in A defining $g(s)$ by (7) with arbitrary v.

An affine representation will be called *reducible* if it leaves a plane W invariant. If we take an affine reper such that the origin and first basic vectors lie in W then

$$(8) \qquad g^i(s) = F^i_j(s) = 0 \quad \text{for} \quad i > p,\ j \leqslant p$$

where $p = \dim W$.

Let W be a plane spanned on all the images of the origin by the transformations (1) of a given affine representation. For simple geometrical reasons W is the minimal invariant plane of this representation. Thus the following proposition is valid only for non-trivial representation.

Proposition 1. *An affine representation is reducible iff it leaves the origin in a proper plane.*

Representations which are not reducible will be called *irreducible.*

Proposition 2. *If the representation $\{F(s)\}$ is irreducible (in the known sense) then the same is true for any affine representation $\{F(s), g(s)\}$.*

The inverse statement is not true. For example, let G be the additive group of real numbers and let

$$F(s) = \begin{bmatrix} 1 & s \\ & 1 \end{bmatrix}, \quad g(s) = \begin{bmatrix} \frac{1}{2} s^2 \\ s \end{bmatrix}.$$

This affine representation is irreducible, although $\{F(s)\}$ is reducible.

Proposition 3. *A non-trivial affine representation $\{F(s), g(s)\}$ is irreducible iff the functions $g^i(s)$ $(i = 1, \ldots, n)$ are linearly independent.*

In fact, if a non-trivial equality

$$\sum_i c_i g^i(s) = d$$

holds for all $s \in G$, then the images of the origin lie in the hyperplane $\sum_i c_i x^i = d$.

The rest follows from proposition 1.

Thus if $\{F(s)\}$ is irreducible then the function components of $g(s)$ are either zeros or linearly independent.

An affine representation will be called *decomposible* if its linear part is such a one. Let $\{F(s)\}$ be decomposible in a direct sum

$$F(s) = \bigcup_{\alpha} F_\alpha(s) = \begin{bmatrix} F_1(s) & & \\ & F_2(s) & \\ & & \ddots \end{bmatrix}.$$

Dividing $g(s)$ in corresponding parts

$$g(s) = \begin{bmatrix} g_1(s) \\ g_2(s) \\ \cdot \\ \cdot \end{bmatrix},$$

we get immediately from (2) and (3)

$$F_\alpha(st) = F_\alpha(s) F_\alpha(t),$$
$$g_\alpha(st) = F_\alpha(s) g_\alpha(t) + g_\alpha(s)$$

and $F_\alpha(1) = E_\alpha$. So we get the decomposition of ther representation $\{F(s), g(s)\}$ in the direct sum of the affine representations $\{F_\alpha(s), g_\alpha(s)\}$, acting in the corresponding planes X_α passing through the origin and invariant under $\{F(s)\}$. Thus we can deal only with non-decomposible representations, and we shall assume that in the sequel.

Now, we are going to present some triviality criteria.

Let $\{F(s)\}$ be a linear representation of the group G; we want to find all vector functions $g(s)$ on G such that $\{F(s), g(s)\}$ is an affine representation of this group. This is the problem of determining all solutions of the functional equation (3) for unknown vector-function $g(s)$, with given $F(s)$ satisfying (2).

As we shall see, such an affine extension of a given linear representation is trivial for a wide class of such representations.

First, notice that any affine representation of degree n can be considered as a linear representation of degree $n+1$; namely, by the homomorphism

(9) $$s \to \tilde{F}(s) = \left[\begin{array}{c|c} F(s) & g(s) \\ \hline 0 & 1 \end{array}\right] \quad (s \in G).$$

One checks easily that $\{\tilde{F}(s)\}$ is a linear representation, i.e. $\tilde{F}(s)$ satisfies the group condition (2) and $\tilde{F}(1) = E$. (From (3) we get $g(1) = 0$). Thus we have the following.

Lemma 1. *All affine representations of a group C are defined by linear representations of this group which are of the form* (9).

Proposition 4. *If for a group G the complete reducibility theorem* * *holds then every affine representation of this true group is trivial.*

Proof. Let a given affine representation $\{F(s), g(s)\}$ be presented in the form (9). By the complete reducibility theorem, $\{F(s)\}$ decomposes into a direct sum of a number of irreducible representations and consequently $\{F(s), g(s)\}$ does also so. Thus we can assume that already $\{F(s)\}$ is irreducible.

* This means that any reducible linear representation of G is fully reducible.

The representation $\{\widetilde{F}(s)\}$ defined by (9) is reducible. Making use of the same theorem, we can find a basis in which $\widetilde{F}(s)$ has the form

$$\widetilde{F}_0(s) = \left[\begin{array}{c|c} F(s) & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{10}$$

This means that there exists a non-singular matrix C such that

$$\widetilde{F}(s) = C^{-1}\widetilde{F}_0(s)C\,, \quad \text{i.e.} \quad C\widetilde{F}(s) = \widetilde{F}_0(s)C \tag{11}$$

for all s. Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

be divided in submatrices like $\widetilde{F}(s)$. By (9) and (10), the second of equalities (11) gives

$$C_3 g(s) = 0\,, \quad C_1 F(s) = F(s) C_1\,, \tag{12}$$

$$C_1 g(s) + C_2 = F(s) C_2\,. \tag{13}$$

By proposition 3, from the first of equalities (12) we get $C_3 = 0$, since the representation is irreducible. Thus C_1 is non-singular and commutes with $F(s)$; from (13) we obtain

$$g(s) = (F(s) - E)v$$

where $v = C_1^{-1}C_2$.

Corollary 1. *Any affine representation of a finite group is trivial.*

This follows from the fact that for finite groups the complete reducible theorem holds true. As we know, this theorem holds also for continuous representations of compact Lie groups and semi-simple Lie groups (see [2]). We can therefore extend this corollary to such affine representations of these groups.

Proposition 5. *If there exists an element $t \in G$ commuting with all elements of G and such that*

$$\det(F(t) - E) \neq 0\,, \tag{14}$$

then any affine representation $\{F(s), g(s)\}$ is trivial.

Proof. From $g(st) = g(ts)$ we get, in view of (3),

$$F(s)g(t) + g(s) = F(t)g(s) + g(t)$$

and hence

$$(F(t) - E)g(s) = (F(s) - E)g(t)\,. \tag{15}$$

From (2) we get that $F(s)$ and $F(t)$ commute, since s and t do so. By (14) the matrix $(F(t)-E)^{-1}$ exists and commutes with $F(s)-E$. Therefore, we can count out $g(s)$ from (15) and get (7) by putting $v = (F(t)-E)^{-1}g(t)$.

The next proposition gives some criteria of triviality of affine representations of the full linear group $GL(n)$. Here $s, t, \dots$ are $n \times n$ non-singular matrices with elements from a field K. By r_p, t_{pq} we denote the following elementary matrices

$$r_p = p\begin{bmatrix} 1 & & p & & \\ & \ddots & \vdots & & \\ \dots & & \varrho & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \qquad t_{pq} = q\begin{bmatrix} 1 & & p & & \\ & \ddots & \vdots & & \\ \dots & & \varrho & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix} \tag{16}$$

where ϱ is a non-vanishing constant. $\{F(s)\}$ denotes a given representation of the linear group. We omit simple proofs of the following two lemmas.

Lemma 2. *If* $g(s) = (F(s)-E)v$ *and* $g(t) = (F(t)-E)v$, *then also* $g(st) = \big(F(st)-E\big)v$.

Lemma 3. *If* $g(s) = (F(s)-E)v$, $\det(F(s)-E) \neq 0$ *and* s, t *commute, then also* $g(t) = \big(F(t)-E\big)v$.

Proposition 6. *If at least one of the following conditions is satisfied, then any affine representation of the group* $GL(n)$ *with the given linear part* $F(s)$ *is trivial*:

(i) *there is exists a scalar matrix* t *such that inequality* (14) *holds*;

(ii) *inequality* (14) *holds for an elementary matrix* r_p (*p-fixed*), *with arbitrary* $\varrho \neq 0$, *and* $n \geqslant 3$;

(iii) $\{F(s)\}$ *is non-decomposible and* $|\det F(s)| \not\equiv 1$.

Proof. Condition (i) is sufficient by proposition 5.

(ii) If s and t commute, then we have equality (15). Substituting there $s = r_q$, $t = r_p$ we get analogously as in the proof of proposition 5:

$$\big(F(r_p)-E\big)g(r_q) = \big(F(r_q)-E\big)g(r_p)$$

with $v = \big(F(r_p)-E\big)^{-1} g(r_p)$. Thus for every elementary matrix of the first type, $g(s)$ is defined by (7). Let t_{pq} be an elementary matrix of the second kind. For $n \geqslant 3$ there exists a matrix r_u such that t_{pq} and r_u commute. Moreover, $\det\big(F(r_u)-E\big) \neq 0$, since it holds for r_p, the matrices r_p and r_u are similar for the same ϱ (see (16)) and so their images $F(r_p)$ and $F(r_u)$ are. From lemma 2 it follows that $g(t_{pq})$ is also defined by (7).

Thus, for any elementary matrix, $g(s)$ has the form (7) with the same v. Any non-singular matrix $s \in GL(n)$ is product of a finite number of elementary matrices (16). Making use of lemma 1, we conclude that $g(s)$ is defined by (7) for any s.

(iii) By assumptions, there exists a matrix t such that $|\det|t| \neq 1$. As it has been said above, t is a product of elementary matrices. Thus the determinant of one of the factor matrices of t must be different from the unit. This elementary matrix cannot be of the second kind. In fact, any two such matrices are similar; in particular, it holds for s and s^2 ($s = t_{pq}$). Hence their images $F(s)$ and $F(s^2)$ are similar and thus they have the same determinant d; from $F(s^2) = F^2(s)$ we get $d = d^2$ and hence $d = 1$. Consequently, the decomposition of t contains a matrix of the type r_p for which $|\det(r_p)| \neq 1$.

Let $\det(r_p) = c$. Since all matrices r_q $(q = 1, \ldots, n)$ with the same ϱ are similar, so, from the analogous reason as above, their images have the same determinant equal c. The matrix $\sigma = r_1, \ldots, r_n = \{\varrho, \ldots, \varrho\}$ is scalar. We have

$$\det F(\sigma) = \det(r_1), \ldots, \det(r_n) = c^n \neq 1$$

from what we conclude that $F(\sigma)$ can not have all eigenvalues equal 1.

On the other hand, σ being a scalar matrix, $F(\sigma)$ commutes with all matrices of the family $\{F(s)\}$. Since this family is non-decomposible, $F(\sigma)$ cannot have an eigenvalue equal 1 and some another ones (see Gantmacher [1], p. 208). Thus the unit does not belong to the spectrum of $F(\sigma)$. Consequently, $\det(F(\sigma) - E) \neq 0$ and our statement holds in virtue of (i). The theorem is proved.

For example, if the linear part of (1) is a tensor transformation with the valence (p, q), and $p \neq q$ then there exists no non-trivial extension to an affine representation. Here the case (i) holds.

If $F(s) = (\det(s))^{-1/n} s$ $(n \geqslant 3)$ then the condition (ii) is fulfilled and $g(s)$ must be trivial.

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