

Arrigo Cellina

On Mappings Defined by Differential Equations *

1. Given the multi-valued differential equation

$$(ME) \quad \dot{x} \in F(t, x)$$

where $F: R^1 \times R^p \rightarrow 2^{R^1}$ is upper semi-continuous and convex, we can define $\pi(\xi, \tau; T)$ to be $\{\eta \in R^p : \exists x(\cdot)$ solution of (ME) such that $x(\tau) = \xi, x(T) = \eta\}$. This is a mapping from R^p into the set of subsets of R^p ; restricting our attention to those values of ξ, τ, T such that $\pi(\xi, \tau, T)$ is not empty, it follows from known theorems that it is a closed multi-valued mapping, i.e. that its graph is a closed subset of $R^p \times R^p$.

It is of some interest to investigate whether this mapping has the property of having a fixed point when mapping a compact convex set of R^p into itself. If this is the case in fact we can have results about the existence of periodic solutions or of critical points. When the function $F(t, x)$ is single-valued and smooth enough to provide uniqueness of solutions, the Schauder Fixed Point Theorem can be used and the existence of periodic solutions or of critical points for ordinary differential equations proved [3], [4]. On the other hand when F is only assumed to be upper semi-continuous (or continuous if single-valued), the mapping π is in general multi-valued; moreover images of points need not be acyclic. It is known that for multivalued mappings with images of points not be acyclic, there need not be fixed points. The point of this note is to show that for the mapping π defined by a multi-valued differential equation (and in particular by an ordinary differential equation) when F is assumed to be upper semi-continuous (in particular continuous) indeed there are fixed points.

In the proof we shall use a generalization of the Schauder Fixed Point Theorem (Theorem 1) that is of interest in itself.

2. For X a metric space and $x, y \in X$, $d(x, y)$ is the distance between x and y and for any $K \subset X$, $d(x, K) = \inf\{d(x, y) : y \in K\}$. For $A, B \in X$ we set $d^*(A, B) = \sup\{d(a, B) : a \in A\}$. We define also an open ball of radius r about x , $B[x, r]$,

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