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On Commuting Linear Operators in a Real Vector Space

The object of this note is to study some properties of a family of linear commuting operators in a real vector space X of finite dimension. Our main aim is to prove a theorem about the existence of a basis in X in which all the matrices of the family are quasi upper triangular. It is a generalization of an analogous well-known theorem in the complex case. The knowledge of the structure of commuting linear operators in the real case is of great importance in many applications, for instance in matrix functional equations.

Since in a fixed basis in X linear operators are uniquely defined by real $n\times n$ matrices ($n = \dim X$), we can use the terms *linear operator* and *matrix* as synonyms. Theorems proved for operators are also valid for real square matrices, and conversely.

1. ALMOST COMPLEX STRUCTURE OF REAL OPERATORS

Let C denote the full matrix representation of the algebra of complex numbers over R. C consists of 2×2 matrices of the form

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} (x, y \in R). \tag{1}$$

We say that a real matrix is *almost complex* if it is an overmatrix (A_{ij}) where A_{ij} are 2×2 submatrices of the form (1), i.e. $A_{ij} \in C$. A linear operator $X \to X$ has the *complex structure* if there exists a basis in X such that its matrix is almost complex.

The algebra $C(2m)$ of all almost complex $2m\times 2m$ matrices is a real representation of the full algebra of complex $m\times m$ matrices and they are isomorph. The mapping $C(2m) \in A = (A_{ij}) \to \widetilde{A} = (z_{ij})$ defined by

$$A_{ij} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \to z_{ij} = x + iy \tag{2}$$

provides a natural isomorphism between them. So we can state the following.

- **Proposition 1.** *The family of almost complex matrices has analogous algebraic properties as the corresponding family of complex ones.*

Let an operator $A: X \to X$ have complex structure and let $\{e_i\}$ $(i = 1, \dots, 2n)$ be a basis in X such that its matrix is almost complex. Define a new complex basis $\{f_i\}$ in X as follows

$$f_k = \frac{1}{\sqrt{2}}(e_{2k-1} - ie_{2k}) \qquad (i = \sqrt{-1};\ k = 1, \dots, n).$$

$$f_{n+k} = \frac{1}{\sqrt{2}}(e_{2k-1} + ie_{2k}).$$

In this basis the matrix of A has the following structure

$$A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix} \tag{3}$$

where $\tilde{A}$ is the image of A in the mapping (2). The subspace $\tilde{X}$ spaned on the vectors $f_1, \dots, f_n$ is invariant for A and the operator $\tilde{A}$ has analogous algebraic properties in $\tilde{X}$ as A in X. For instance, if $\tilde{A}$ is upper triangular, then A is also such one as an overmatrix with elements from C, i.e. A_{ij} are zero matrices for $i > j$.

From (3) we conclude easily that an operator has complex structure if and only if the sequence of its elementary divisors consists of the pairs

$$(\lambda - \lambda_0)^k, (\lambda - \bar{\lambda}_0)^k$$

where λ_0 may be real. In particular, any real operator having only complex eigen-values has complex structure.

Proposition 2. *Let F be a family of real commuting matrices. If a matrix $A \in F$ has the complex numbers $\lambda, \bar{\lambda}$ $(\lambda = a + bi,\ b \neq 0)$ as unique eigen-values, then in a suitable basis all matrices of F are almost complex.*

Proof. A basis can be choosen so that A has its real canonical Jordan form. Then A is quasidiagonal with the blocks $L_1, \dots, L_t$ where

$$L = \begin{bmatrix} J & E & \\ & J \ddots & E \\ & & \ddots J \end{bmatrix} \tag{4}$$

with $J = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ and $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Evidently A is almost complex. Let $B \in F$. According to the quasidiagonal form of A we divide B in blocks $B = (B_{rs})$, $r, s = 1, \dots, t$. From the equality $AB = BA$ we get

$$L_r B_{rs} = B_{rs} L_s. \tag{5}$$

We divide B_{rs} in 2×2 blocks

$$B_{rs} = (X_{ij}) \qquad (i = 1, \dots, p;\ j = 1, \dots, q)$$

where $2p = \dim L_r$ and $2q = \dim L_s$. Taking into account the special structure of the matrices L_r and L_s defined by (4) we get immediately

$$JX_{ik} - X_{ik}J = X_{i+1,k} - X_{i,k-1} \tag{6}$$

putting $X_{p+1,k} = X_{i0} = 0$.

Our object is to show that any 2×2 matrix X_{ik} has the form (1), i.e. $X_{ik} \in C$. We will do it by induction with respect to the index difference $k-i$ increasing from $1-p$ to $q-1$. For X_{p1} we get from (6)

$$JX_{p1} - X_{p1}J = 0$$

what easily implies $X_{p1} \in C$. Assume $X_{jl} \in C$ for any pair (j, l) such that $l-j \leqslant m$. Let $k-i = m+1$. Then the matrices standing on the right of (6) have their index difference equal to m, so they belong both to C. Hence

$$JX_{ik} - X_{ik}J \in C. \tag{7}$$

From (7) we get easily $X_{ik} \in C$. Our statement is proved. Thus any matrix B_{rs} and consequently the whole B consists of 2×2 submatrices belonging to C, so that B is almost complex.

Obviously, the above proposition holds also if the family F contains a matrix having only complex eigen-values (not necessarily one pair). This follows from the fact that if A is a direct sum $L+M$, i.e.

$$A = \begin{bmatrix} L & \\ & M \end{bmatrix},$$

and L, M have different eigen-values then the same structure has [1] any matrix B commuting with A, i.e.

$$B = \begin{bmatrix} B_1 & \\ & B_2 \end{bmatrix}, \quad \dim B_1 = \dim L, \ldots$$

2. TRIANGULAR FORM OF REAL OPERATORS

Theorem. *Let X be a real vector space of finite dimension and let F be a family of commuting linear operators $X \to X$. Then there exists a basis in X and a decomposition of X into a direct sum,*

$$X = X_1 + \ldots + X_s, \tag{8}$$

such that

(*i*) *every subspace X_k is invariant for any $A \in F$,*

(*ii*) *the minimal polynomial of X_k with respect to any $A \in F$ is a degree of an irreducible real polynomial,* *(iii) for any $A \in F$ its matrix is either upper triangular or it is almost complex and is upper triangular as an overmatrix with 2×2 matrix elements belonging to C.*

Proof. We have to prove only statement (iii) because the first two express the well-known fact which holds for any family of commuting operators in a real vector space ([1], p. 206). If all the operators of the family F restricted to a subspace X_k have only real eigen-values then such a basis may be choosen that their matrices are upper triangular. In order to prove this we can follow the same procedure as in the complex case [4], [3]; it is based on the fact that any two complex operators have at least one common eigen-vector.

Assume that there exists an operator $A \in F$ which restricted to X_k has a pair of complex conjugate eigen-values. In view of (ii) this pair is unique. According to proposition 2 its matrix and the matrices of all the operators of F restricted to X_k are almost complex in a suitable basis. Thus we can consider the corresponding complex family $\tilde{F}$ of operators in $\tilde{X}_k$. In virtue of the Morozoff-Kurepa theorem a basis can be choosen in $\tilde{X}_k$ such that the matrices of $\tilde{F}$ are upper triangular. Passing by the inverse to the isomorphism (2) to the real matrices of F in X_k we get the quasitriangular form stated in (iii).

We shall say that a real matrix is *almost diagonal* if it is quasidiagonal and has on its diagonal the scalars or the 2×2 blocks of the form (1), i.e. a matrix

$$\left\{d_1, \ldots, d_s, \begin{bmatrix} a_1 & b_1 \\ -b_1 & a_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_t & b_t \\ -b_t & a_t \end{bmatrix}\right\}, \quad b_i \neq 0.$$

Proposition 3. *If F is a family of real commuting and non-singular matrices then in a suitable basis any matrix $A \in F$ can be presented in the form*

$$A = DA_1 = A_1 D\,* \tag{9}$$

where D is almost diagonal and A_1 is upper triangular with the 1's on the main diagonal.

In fact, by theorem 1 there exists a basis such that the matrices of F have a quasi-triangular form described there. By excluding the quasidiagonal non-singular matrix D consisting of the elements standing on the diagonal we get the required decomposition (9). Obviously, the almost diagonal matrix D commutes with the almost complex matrix A_1.

Proposition 4. *Let F be a family of real (complex) matrices satisfying the matrix equation*

$$X^p = \varepsilon E \quad (\varepsilon^2 = 1,\ p \text{ integer} \neq 0). \tag{10}$$

A basis can be found such that all the matrices of F are almost diagonal (diagonal) if and only if they commute.

In fact, it is necessary because any two almost diagonal matrices commute. Conversely, if they commute, then in a basis they can be presented in the form (9),

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* This decomposition was given (without proof) by the author and M. Kuczma in [2].

where D is diagonal in the complex case. Since D^p is also almost diagonal and A_1^p has only the 1's on the main diagonal, so

$$D^p = \varepsilon E \quad \text{and} \quad A_1^p = E. \tag{11}$$

Taking into account the spacial form of A_1 we get from the second of equations (11) that $A_1 = E$, what completes the proof.

For $p = 2$ and $\varepsilon = -1$ we get a unique solution for D satisfying (11), which in a suitable basis has the form

$$D = \left\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right\}.$$

3. INVARIANT SUBSPACES

A. Let F be a family of commuting linear operators $X \to X$ such that any operator of F has a unique eigen-value equal 1. Denote by X_1 the subspace of all common eigen-vectors of the family,

$$X_1 = \{x : Ax = x \text{ for any } A \in F\}.$$

X_1 is not empty for commuting operators. Let

$$X_2 = \{x : Ax = x (\operatorname{mod} X_1) \text{ for any } A \in F\}.$$

We have $X_1 \neq X_2$ because X_1 is invariant and the operators of F restricted to any complementary of X_1 are also commuting and there exists a common eigen-vector $\operatorname{mod} X_1$, not belonging to X_1. In this way we define a sequence of subspaces

$$X_1 \subset X_2 \subset \dots \subset X_s = X, \qquad X_i \neq X_k \text{ if } i \neq k \tag{12}$$

which are invariant for any $A \in F$.

If we choose a basis $\{e_i\}$ in X such that for any X_k the corresponding sequence $e_1, \dots, e_{ak}$ of successive vectors of the basis forms a basis in X_k, then any operator of F will have a matrix of the form

$$\begin{bmatrix} E_1 X_{12} \dots & & \\ & E_2 X_{23} \dots & \\ & \ddots & \\ & & E_s \end{bmatrix}. \tag{13}$$

Here E_i are unit matrices of corresponding dimensions and for any fixed i the matrices $X_{i,i+1}$ have no common zero-space, i.e. there does not exist a constant vector $\xi \neq 0$ such that $X_{i,i+1}\xi \equiv 0$.

B. Let A be an operator with a real minimal polynomial φ. If

$$\varphi = \varphi_1 \dots \varphi_t$$