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On Semigroups of Linear Operators

Let Q be an arbitrary additive semigroup of rational numbers containing the unit. Let X be an n-dimensional vector space over a field K and let F be a representation of Q in the semigroup of all transformations of X with multiplication as a binary operation. We have

$$F(r+r') = F(r)F(r') \quad (r, r' \in Q). \tag{1}$$

We assume that $\{F(r)\}$ is a regular representation, i.e. $F(r)$ is regular for any $r \in Q$. For a fixed basis of X, $F(r)$ can be considered as an $n\times n$ matrix with elements from the field K. Then (1) becomes a matrix functional equation *.

S. Kurepa [5] and D. Z. Djoković [1] have proved that if Q is the semigroup of rational numbers of the form $l/2^k$ $(l, k = 0, 1, 2, \ldots)$ then in a suitable basis any complex solution of (1) has the form

$$F(r) = U(r)\exp(rC) = \exp(rC)\,U(r) \tag{2}$$

where $U(r)$ is a semigroup of diagonal unitary matrices and C is the sum of a nilpotent and a real diagonal matrix.

In this note we will generalize this result for any semigroup Q mentioned above (in particular for the full additive group of rational numbers) and for the case where K is the field of real numbers.

Lemma 1. *Any non-vanishing solution of the functional equation*

$$z(r+r') = z(r)z(r') \quad (r, r' \in Q) \tag{3}$$

is of the form

$$z(r) = u(r)\exp(ra), \tag{4}$$

if $z(r)$ is complex, and of the form

$$z(r) = \exp(ra), \tag{5}$$

if $z(r)$ is real (a is a real constant and $|u(r)| = 1$).

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* In the common papers [2], [3] and [4] M. Kuczma and the author have considered the equation $F(rr') = F(r)F(r')$ (r, r' real) under some additional conditions.

Proof. For every r, $z(r) \neq 0$, otherwise we would have $z(r) \equiv 0$. Moreover, if $z(r)$ is real then $z(r) = z(r/2)z(r/2) > 0$. Thus in both cases we can write $z(r) = \exp w(r)$. From (3) we get

$$\exp w(r+r') = \exp\left(w(r)+w(r')\right)$$

and hence

$$w(r+r') = w(r)+w(r')(\operatorname{mod} 2\pi i)\,.$$

The real part of $w(r)$ satisfies the Cauchy additive equation and thus we have for r rational

$$Re\, w(r) = ra\,, \qquad a \in R$$

and consequently we get

$$w(r) = ra + v(r)i \tag{6}$$

where $v(r)$ is real. For $z(r)$ real, $v(r) = 0$. This gives (4) with $u(r) = \exp\left(v(r)i\right)$, and (5) in the real case.

By $[A, B] \neq 0$ we denote the commutator of A and B. By I we denote the set of all matrices which have a unique eigen-value equal 1. The following lemma is valid for complex as well as for real matrices.

Lemma 2. *If $A \in I$ then the equation*

$$X^p = A\,, \quad (p\text{-integer} > 1) \tag{7}$$

has only one solution B such that $[A, B] = 0$ and $B \in I$. This solution is defined by

$$B = \exp\left(M\frac{1}{p}\right) \tag{8}$$

where

$$M = \operatorname{Log} A = (A-E) - \frac{1}{2}(A-E)^2 + \ldots + \frac{(-1)^{k+1}}{k}(A-E)^k \tag{9}$$

with k such that $(A - E^{k+1}) = 0$.

Proof. Since M defined by (9) is nilpotent, B defined by (8) belongs to I and is a polynomial in M and consequently in A too. Thus $[A, B] = 0$. Evidently B satisfies equation (7). Let C be a matrix satisfying (7) such that $[A, C] = 0$ and $C \in I$. Then we have $[B, C] = 0$ and $B^p = C^p$. Since B and C^{-1} commute, we have $E = B^p C^{-p} = (BC^{-1})^p$ and thus

$$(BC^{-1})^p = E \quad (p > 1)\,. \tag{10}$$

Since $B \in I$, $C^{-1} \in I$ and the matrices commute, also

$$BC^{-1} \in I\,. \tag{11}$$

From (10) and (11) we conclude easily that $BC^{-1} = E$ *, and thus $B = C$.

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* This is easily seen if we take a matrix fulfiling (10) and (11) in its canonical form.

Lemma 3. *If $F(r)$ satisfies* (1) *and $F(r) \in I$ for any $r \in Q$, then*

$$F(r) = \exp(rM) \tag{12}$$

where M is a nilpotent matrix.

Proof. Let us put $A = F(1)$ and let p be any integer > 0. By (1) we have

$$A = [F(1/p)]^p .$$

Thus $F(1/p)$ satisfies equation (7) and moreover it fulfills both conditions of lemma 2. We have therefore $F(1/p) = \exp\left(M\frac{1}{p}\right)$ where M is nilpotent. For a rational number $r = p/q$ we get hence

$$F(r) = \exp(Mp/q) .$$

Lemma 4. *Let*

$$A(r) = \begin{bmatrix} a(r) & b(r) \\ -b(r) & a(r) \end{bmatrix}$$

be a real matrice defined for $r \in Q'$ and fulfilling equation

$$A(r+r') = A(r)A(r') . \tag{13}$$

Then

$$A(r) = \exp(ra)\cdot\begin{bmatrix} \cos\beta(r)\sin\beta(r) \\ -\sin\beta(r)\cos\beta(r) \end{bmatrix} = \exp\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \tag{14}$$

where a, b are constants and $\beta(r) = rb (\operatorname{mod} 2\pi)$.

In fact, equation (13) is equivalent to (3) by the mapping

$$z(r) = a(r) + b(r)i \to A(r)$$

defining the matrix representation of the algebra of complex number over R. From (4) we get (14) where $\beta(r)$ must satisfy the equality

$$\beta(r+r') = \beta(r) + \beta(r') (\operatorname{mod} 2\pi)$$

from which we get, for $r \in Q$, $\beta(r) = rb (\operatorname{mod} 2\pi)$.

We will say that a matrix D is *almost diagonal* if it is quasidiagonal and if its elements on the main diagonal are scalars or 2×2 matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Theorem. *If $\{F(r)\}$ is a regular complex representation of a semigroup Q of rational numbers then in a suitable base $F(r)$ is defined by* (2) *and if $F(r)$ is real then it has the form*

$$F(r) = D(r)\exp(Mr) = \exp(Mr)D(r)$$

where M is a nilpotent matrix and $D(r)$ is, in general, an almost diagonal matrix with elements of the form (5) *or* (14).

Proof. In view of (1) $\{F(r)\}$ is a commutative family. As it was shown in [6] there exists a basis of X such that the matrices $F(r)$ are simultaneously decomposible as follows

$$F(r) = D(r)F_1(r) = F_1(r)D(r)$$

where $F_1(r) \in I$ and is upper triangular, and $D(r)$ is diagonal in the complex case and diagonal or almost diagonal if $F(r)$ is real.

From the special forms of matrices D and F_1 it follows that they satisfy both (1) if F does so. If $F(r)$ is complex then $D(r)$ has on its diagonal the functions (4) and $F_1(r)$ has the form (12). From this we obtain the form (2) by an easy transformation. Analogously we prove our statement in the real case taking into account lemmas 1, 3 and 4.

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