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## Inequalities of Wirtinger's Type and Their Discrete Analogues

In this note we present some inequalities of Wirtinger type for functions defined on an interval  $[0, h]$ . These results have exact discrete analogues that we formulate and prove in the third section of the paper. In the fourth section we state a theorem concerning the convergence of the coefficient in a particular inequality for discrete case to the coefficient in the suitable inequality for continuous case.

The inequalities obtained in this note, as well for continuous case as for discrete one, are closely related to the boundary-value problems for differential and difference equations respectively (see [2]).

### 1. NOTATIONS AND PRELIMINARIES

Let  $R^m$  be the  $m$ -dimensional real Euclidean space with the usual scalar product of vectors  $p_1, p_2$  denoted by  $\langle p_1, p_2 \rangle$  and with the Euclidean norm  $|p| = \sqrt{\langle p, p \rangle}$ . By  $L^2([0, h])$  we denote the space of all square sommable functions on  $[0, h]$  with the scalar product and norm defined, respectively, by the formulae

$$\langle y_1, y_2 \rangle = \int_0^h y_1 y_2 dt, \quad \|y\| = \sqrt{\langle y, y \rangle}.$$

The set  $\{0, \dots, n\}$  will be denoted by  $N$ . The difference operators  $\Delta^{(k)}: R^{n+1} \rightarrow R^{n+1}$ ,  $\nabla^{(k)}: R^{n+1} \rightarrow R^{n+1}$  for  $k \in N$  are defined by the formulae

$$\Delta^{(k)}v = (\Delta^{(k)}v_0, \dots, \Delta^{(k)}v_n), \quad \nabla^{(k)}v = (\nabla^{(k)}v_0, \dots, \nabla^{(k)}v_n),$$

where for  $i \in N$  we set

$$\begin{aligned} \Delta^{(0)}v_i &= v_i, & \nabla^{(0)}v_i &= v_i \\ \Delta v_i &= \Delta^{(1)}v_i = \begin{cases} v_{i+1} - v_i, & i = 0, \dots, n-1 \\ 0, & i \in n, \end{cases} & \nabla v_i &= \nabla^{(1)}v_i = \begin{cases} 0, & i = 0 \\ v_i - v_{i-1}, & i = 1, \dots, n, \end{cases} \end{aligned}$$

and, for an integer  $k$  satisfying inequality  $2 \leq k \leq n$ ,

$$\Delta^{(k)}v_i = \begin{cases} \Delta(\Delta^{(k-1)}v_i), & i = 0, \dots, n-k \\ 0 & , i = n-k+1, \dots, n \end{cases} \quad \nabla^{(k)}v_i = \begin{cases} 0 & , i = 0, \dots, k-1 \\ \nabla(\nabla^{(k-1)}v_i), & i = k, \dots, n. \end{cases}$$

Composing several times in arbitrary succession the difference operators defined above we get so-called mixed difference operators. For example, if  $\nu$  is an odd integer, and  $k_1, \dots, k_\nu$  are integers such that

$$k_2 + k_4 + \dots + k_{\nu-1} \leq n - (k_1 + k_3 + \dots + k_\nu),$$

then for  $i \in N$  we have

$$\Delta^{(k_\nu)} \nabla^{(k_{\nu-1})} \dots \nabla^{(k_2)} \Delta^{(k_1)} v_i = \begin{cases} 0 & , i = 0, \dots, k_2 + k_4 + \dots + k_{\nu-1} - 1, \\ & n - (k_1 + k_3 + \dots + k_\nu) + 1, \dots, n \\ \Delta(\Delta^{(k_\nu-1)} \nabla^{(k_{\nu-1})} \dots \nabla^{(k_2)} \Delta^{(k_1)} v_i), & i = k_2 + k_4 + \dots + k_{\nu-1}, \dots \\ & \dots, n - (k_1 + k_3 + \dots + k_\nu). \end{cases}$$

From these definitions we obtain immediately the following formulae

$$(1.1) \quad \Delta^{(k)}v_i = \begin{cases} \sum_{\nu=0}^k (-1)^{k-\nu} \binom{k}{\nu} v_{i+\nu}, & i \geq 0, \dots, n-k \\ 0 & , i = n-k+1, \dots, n \end{cases}$$

$$(1.2) \quad \nabla^{(k)}v_i = \begin{cases} 0 & , i = 0, \dots, k-1 \\ \sum_{\nu=0}^k (-1)^\nu \binom{k}{\nu} v_{i-\nu}, & i = k, \dots, n \end{cases}$$

$$(1.3) \quad \sum_{i=0}^n u_i \cdot \Delta v_i + \sum_{i=0}^n \nabla u_i \cdot v_i = u_n v_n - u_0 v_0.$$

In the sequel formula (1.3), as the discrete analogue of the integration by parts formula, will be called *summation by parts formula*.

## 2. CONTINUOUS CASE

For an integer  $l$  we denote by  $K_{2l}$  the subspace of  $L^2([0, h])$  consisting of all functions possessing derivatives up to the order  $2l$  and satisfying the condition  $y^{(p)}(0) = y^{(p)}(h) = 0$  ( $p = 0, \dots, l-1$ ). Let  $T: K_{2l} \rightarrow L^2([0, h])$  be an operator defined by the formula

$$(2.1) \quad Ty = (-1)^l \frac{d^{(l)}}{dt^l} y^{(l)}.$$

Theorem 1. For any function  $y \in K_{2l}$  we have the inequality

$$(2.2) \quad \|y\|^2 \leq \frac{1}{\lambda_1} \|y^{(l)}\|^2,$$

where  $\lambda_1$  is the smallest positive number such that the equation

$$(2.3) \quad Ty = \lambda y$$

has nontrivial solutions in  $K_{2l}$ .

Proof. It is well known that the functional  $G: K_{2l} \rightarrow R$  given by the formula

$$G(y) = \|y^{(l)}\|^2$$

attains its minimum in the set of all functions of  $K_{2l}$  satisfying the condition

$$(2.4) \quad \|y\| = 1$$

only for the solutions of equation (2.3). It is also easily seen that for  $y \in K_{2l}$  we have

$$(2.5) \quad G(y) = \langle Ty, y \rangle = \langle y, Ty \rangle.$$

Hence, by (2.3), it follows that the minimum of the functional  $G$  taken over all the functions of  $K_{2l}$  satisfying condition (2.4) is equal to the smallest positive eigen-value of the operator  $T$  (i.e. to a smallest positive number  $\lambda$  such that equation (2.3) has nontrivial solutions). Denoting it by  $\lambda_1$  we get

$$G\left(\frac{y}{\|y\|}\right) \geq \lambda_1$$

for any  $y \in K_{2l}$  such that  $\|y\| > 0$ .

The case  $\|y\| = 0$  is trivial. In the case  $l = 1$  one can easily compute  $\lambda_1$  — it is now the smallest positive number such that there is a nontrivial solution of the differential equation

$$y'' + \lambda y = 0,$$

satisfying the condition  $y(0) = y(h) = 0$ . Since the solution of this equation has the form

$$y(t) = C_1 \cos \sqrt{\lambda} t + C_2 \sin \sqrt{\lambda} t,$$

where  $C_1, C_2$  are arbitrary real numbers, we get  $\lambda_1 = \frac{\pi^2}{h^2}$ . Therefore, inequality (2.2) for  $l = 1$  and  $y \in K_2$  has the form

$$(2.6) \quad \int_0^h y^2 dt \leq \frac{h^2}{\pi^2} \int_0^h (y')^2 dt.$$

This is the integral form of the well-known inequality of Wirtinger (see [1], [3]).

Similarly, for  $l = 2$  it is easy to compute  $\lambda_1$ . In this case  $\lambda_1$  is the smallest positive number such that the differential equation

$$y^{(iv)} - \lambda y = 0$$

has a nontrivial solution satisfying the condition

$$y(0) = y'(0) = y'(h) = y(h) = 0.$$

The solution of this equation has the form

$$y(t) = C_1 e^{\sqrt[4]{\lambda}t} + C_2 e^{-\sqrt[4]{\lambda}t} + C_3 \cos \sqrt[4]{\lambda}t + C_4 \sin \sqrt[4]{\lambda}t,$$

where  $C_i$  ( $i = 1, \dots, 4$ ) are arbitrary real numbers.

Hence and from the above condition it follows the  $\lambda_1$  is the smallest root of the equation

$$(2.7) \quad ch(\sqrt[4]{\lambda}h) \cos(\sqrt[4]{\lambda}h) - 1 = 0,$$

which gives

$$\lambda_1 = \left( \frac{3\pi + 2\varepsilon}{2\pi} \right)^4 \frac{\pi^4}{h^4},$$

where  $\varepsilon$  is the positive number such that  $z_1 = \frac{3}{2}\pi + \varepsilon$  is the smallest positive root of the equation

$$ch z = \sec z.$$

Thus, for  $l = 2$  and  $y \in K_4$ , we can write inequality (2.2) in the following integral form

$$(2.8) \quad \int_0^h y^2 dt \leq \left( \frac{2\pi}{3\pi + 2\varepsilon} \right)^4 \frac{h^4}{\pi^4} \int_0^h (y'')^2 dt < \left( \frac{2h}{3\pi} \right)^4 \int_0^h (y'')^2 dt.$$

The sign  $<$  in (2.8) occurs only in the case  $y \neq 0$ .

### 3. DISCRETE CASE

Now we shall consider discrete analogues of the class  $K_{2l}$  and the operator  $T$ . Namely we set

$$K_{2l}^n = \{v \in R^{n+1} : v_p = v_{n-p} = 0, \quad p = 0, \dots, l-1\}$$

and define the operator  $T_n : K_{2l}^n \rightarrow R^{n+1}$  by the formula

$$(3.1) \quad T_n v = (-1)^l V^{(l)} \Delta^{(l)} v.$$

We suppose that the numbers  $l$  and  $n$  are fixed and fulfil the inequality  $2l \leq n$ . The following theorem is an exact discrete analogue of Theorem 1.

**Theorem 2.** For any  $v \in K_{2l}^n$  we have the inequality

$$(3.2) \quad |v|^2 \leq \frac{1}{\lambda_1^n} |\Delta^{(l)}v|^2$$

where  $\lambda_1^n$  is the smallest positive number such that the equation

$$(3.3) \quad T_n v = \lambda v$$

has nontrivial solutions in  $K_{2l}^n$ .

**Proof.**  $K_{2l}^n$  as a subspace of the Hilbert space  $R^{n+1}$  is obviously also a Hilbert space. It is easy to see that  $T_n$  is a self-adjoint operator. Indeed, we have the equality

$$\langle T_n v_1, v_2 \rangle = \langle v_1, T_n v_2 \rangle \quad (v_1, v_2 \in K_{2l}^n),$$

which can be easily obtained applying  $l$  times to its left-hand side the summation by parts formula, what really can be done owing to the definitions of our difference operators. In fact, for  $k \in \{0, \dots, l-1\}$  we have

$$\nabla^{(l-k)} \Delta^{(l)} v_i = \begin{cases} \nabla (\nabla^{(l-k-1)} \Delta^{(l)} v_i), & i = l-k, \dots, n-1 \\ 0, & i = 0, \dots, l-k-1, n-l+1, \dots, n. \end{cases}$$

But the assumption  $v \in K_{2l}^n$  implies that  $\Delta^{(k)} v_i \in K_{2(l-k)}^n$  and, therefore,  $\Delta^{(k)} v_i = 0$  ( $i = 0, \dots, l-k-1, n-l+1, \dots, n$ ). Hence, and from the definition of  $\nabla^{(l-k)} \Delta^{(l)} v_i$ , we get the equality

$$\nabla^{(l-k)} \Delta^{(l)} v_i \cdot \Delta^{(k)} v_i = \nabla (\nabla^{(l-k-1)} \Delta^{(l)} v_i) \cdot \Delta^{(k)} v_i \quad (i \in N),$$

which enables us to apply step by step formula (1.3). In a similar way one can obtain

$$(3.4) \quad \langle T_n v, v \rangle = G_n(v), \quad (v \in K_{2l}^n),$$

where the map  $G_n: R^{n+1} \rightarrow R$  is given by the formula

$$G_n(v) = |\Delta^{(l)}v|^2.$$

$T_n$  as a self-adjoint operator has real eigen-values. Hence, by (3.3) and (3.4) we get, for the vectors  $v \in K_{2l}^n$  satisfying the condition  $|v| = 1$ , the equality

$$G_n(v) = \lambda,$$

where  $\lambda$  is an eigen-value of  $T_n$ .

Thereby the minimum of the non-negative function  $G_n$  over the set  $K_{2l}^n \cap S_n$ , where  $S_n = \{v \in R^{n+1} : |v| = 1\}$ , is equal to the smallest eigen-value of the operator  $T_n$ .

From the continuity of the map  $G_n$  and the compactness of the set  $K_{2l}^n \cap S_n$  it follows that the minimum is attained and, therefore, is positive. We denote it by  $\lambda_1^n$ . Thus, for any  $v \in K_{2l}^n$ ,  $|v| > 0$ , we have

$$G_n\left(\frac{v}{|v|}\right) > \lambda_1^n$$

which gives the required inequality (3.2). Since the case  $|v| = 0$  is trivial, the proof is completed.

From the definition of the difference operators (see section 1) it easily follows that for  $v \in K_{2l}^n$  we have

$$|\Delta^{(l)}v| = |\Delta_{s_1}^{(k_1)} \dots \Delta_{s_p}^{(k_p)}v| \quad (k_i > 0, s_i \in \{-1, 1\}, (i = 1, \dots, p), k_1 + \dots + k_p = l)$$

where  $\Delta_{s_1}^{(k_1)} \dots \Delta_{s_p}^{(k_p)}$  denotes the mixed difference operator with  $\Delta_{s_i}^{(k_i)} = \Delta^{(k_i)}$ , if  $s_i = 1$ , and  $\Delta_{s_i}^{(k_i)} = \nabla^{(k_i)}$ , if  $s_i = -1$ . Hence, from Theorem 2 we obtain immediately the following

**Theorem 3.** For any  $v \in K_{2l}^n$  we have the inequality

$$(3.5) \quad |v|^2 \leq \frac{1}{\lambda_1^n} |\Delta_{s_1}^{(k_1)} \dots \Delta_{s_p}^{(k_p)}v|^2,$$

where  $\lambda_1^n$  is the smallest positive eigen-value of the operator  $T_n$  defined by formula (3.1).

For  $l = 1$  one can easily compute  $\lambda_1^n$ . In this case, accordingly to Theorem 2,  $\lambda_1^n$  is the smallest positive number  $\lambda$  such that equation (3.3) has nontrivial solutions in  $K_2^n$ . Equation (3.3) written in the coordinates has the form

$$\nabla \Delta v_i + \lambda v_i = 0 \quad (i \in N),$$

and its general solution is given by the formula

$$v_i = C_1 \cos\left(i \arccos \frac{2-\lambda}{2}\right) + C_2 \sin\left(i \arccos \frac{2-\lambda}{2}\right) \quad (i \in N),$$

where  $C_1, C_2$  are arbitrary constants and where without loss of generality we can assume that the inequality  $0 \leq \lambda \leq 4$  is fulfilled. From the conditions  $v_0 = v_n = 0$ ,  $|v| \neq 0$  we deduce

$$C_1 = 0, \quad C_2 = 0, \quad \sin\left(n \arccos \frac{2-\lambda}{2}\right) = 0$$

and, therefore,  $\lambda_1^n = 4 \sin^2 \frac{\pi}{2n}$ .

Thus, in the case  $l = 1$  inequality (3.2) admits the form

$$(3.6) \quad |v|^2 \leq \frac{1}{4 \sin^2 \frac{\pi}{2n}} |\Delta v|^2 \quad (v \in K_2^n).$$

This is the known discrete analogue of Wirtinger's inequality (see [4], [5]).

