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Remarks on Algebraic Properties of Sequential Machines

INTRODUCTION

The notion of a sequential machine is very important for the theoretic cybernetics. Sequential machines are abstract schemes of many real problems we can find in almost whole human activity, for instance, in the industry, trade, medicine, communication etc. In this note we will not be interested in giving other examples of adaptations of sequential machines to the investigation of new problems. The purpose of this note is to give an algebraic characterization of sequential machines and state their properties from the algebraic point of view. The cardinal purpose of the note is to state relations between the theory of sequential machines and the theory of algebraic objects (considered in [10] and [1]). This problem was suggested by A. Zajtz as a completion of his works in the theory of algebraic objects. The basic concepts of this note were reported at a Student Seminar held in Toruń in March 1966.

Algebraic objects were introduced by A. Zajtz [10] in connection with an investigation of the theory of geometrical objects, more precisely, in connection with investigations of concomitants of given geometrical objects. Algebraic objects are abstract algebraic models of geometrical objects. Moreover, they may serve also as algebraic models of abstract sequential machines [3]. We discuss here this problem.

In Section 1 we formulate the definition of a sequential machine. This definition is similar to the definition (of quasi-machine) given by S. Ginsburg [3] but it is more general (we do not assume, in general, that functions δ_A and λ_A are defined everywhere). We also define here Mealy-Moore sequential machines. In the next sections we formulate a sequence of propositions about sequential machines. Throughout the note we give a lot of examples which illustrate properties of machines, especially negative ones from the algebraic point of view. In all these examples sequential machines are defined by their graphs. This method is used very often and it is precisely exposed, for instance, in [4], [2], [7], [5].

1. BASIC DEFINITIONS

The definition of a sequential (quasi-)machine we introduce below coincides almost exactly with the definition given by S. Ginsburg [3]. Namely, in contradistinction to [3], we do not assume here that the next state function δ_A and the input function λ_A are defined everywhere. Machines with "partially" defined functions δ_A and λ_A were considered in [2], [4], [5], [7], but in those works the authors consider other aspects of the theory of sequential machines as in this note. In [3] and [8] the authors restrict their considerations only to a special kind of machines called in this note complete machines. Before we state precisely the definitions of machines and complete machines we give some remarks about partially defined maps.

Let X and Y be arbitrary sets, and let U be any subset of X . A map $h: U \rightarrow Y$ is called a *partially defined map* of X into Y . In the sequel, the set U will be denoted by D_h . For two partially defined maps $h: D_h \rightarrow Y$ and $g: D_g \rightarrow Z$, where D_h is a subset of X and D_g is a subset of Y , we define the *composition*

$$g \circ h: D_{g \circ h} \rightarrow Z, \quad (g \circ h)(x) = g(h(x)) \quad \text{for } x \in D_{g \circ h},$$

where $D_{g \circ h} = h^{-1}(D_g)$ is a subset of X . In the particular case $D_g \cap h(D_h) = \emptyset$ the composition $g \circ h$ is defined on the empty set (it is the *empty map*).

Definition 1. A *machine* $*A$ is a 5-tuple $(S_A, W_A, U_A, \delta_A, \lambda_A)$ consisting of

- (1) a non-empty set S_A called the *set of states of A*,
- (2) a semi-group W_A (with unit e) called the *output semi-group of A*,
- (3) a semi-group U_A called the *input semi-group of A*,
- (4) a map $\delta_A: D_{\delta_A} \rightarrow S_A$ called the *next state function*, where D_{δ_A} is some subset of $S_A \times W_A$,
- (5) a map $\lambda_A: D_{\lambda_A} \rightarrow U_A$ called the *input function*, where D_{λ_A} is some subset of $S_A \times W_A$,

and satisfying the conditions

- (a) $\delta_A(a, e) = a$ for each state $a \in S_A$,
- (b) δ_A is defined at (a, w) and (b, w') (where $b = \delta_A(a, w)$) if and only if δ_A is defined at (a, ww') , and, whenever these symbols are defined,

$$\delta_A(\delta_A(a, w), w') = \delta_A(a, ww'),$$

- (c) if δ_A is defined at (a, w) and λ_A is defined at (a, w) and $(\delta_A(a, w), w')$ then δ_A is defined at (a, ww') and

$$\lambda_A(a, ww') = \lambda_A(a, w)\lambda_A(\delta_A(a, w), w').$$

* A 5-tuple A having properties (a)–(c) is called by S. Ginsburg [3] a *quasi-machine* (he assume also that δ_A and λ_A are defined everywhere). By a *machine*, he means a quasi-machine in which the input-group U_A satisfies the left cancellation law (i.e., $ww' = ww''$ implies that $w' = w''$). In this note we will not use that property.

Definition 2. A machine A is said to be *complete* if the maps δ_A and λ_A are defined in whole set $S_A \times W_A$ (i.e., $D_{\delta_A} = D_{\lambda_A} = S_A \times W_A$) ([7], p. 46).

Mealy-Moore machines are the most important examples of machines. They were considered in [2], [4], [5], [8], and they can be defined as follows.

Definition 3 ([5], p. 100–101). A *Mealy sequential machine* is a 5-tuple $A = (S_A, X_A, Y_A, \delta_A, \lambda_A)$ consisting of

- (1) a set S_A called the *set of states of A* ,
- (2) sets X_A and Y_A called the *output and the input alphabet of A* ,
- (3) a map $\delta_A: D_{\delta_A} \rightarrow S_A$ called the *next state function*, where D_{δ_A} is some subset of $S_A \times X_A$,
- (4) a map $\lambda_A: D_{\lambda_A} \rightarrow Y_A$ called the *input function*, where D_{λ_A} is some subset of $S_A \times X_A$.

Definition 4 ([5], p. 102). A *Moore machine* is a 5-tuple $A = (S_A, X_A, Y_A, \delta_A, \mu_A)$, where S_A, X_A, Y_A and δ_A are as in Definition 3, and $\mu_A: D_{\mu_A} \rightarrow Y_A$ is a map of some subset D_{μ_A} of S into Y_A .

Each Moore machine can be considered as Mealy one. In order to see this, it is sufficient to put $\lambda_A = \delta_A \circ \pi_A$, where $\pi_A: S_A \times X_A \rightarrow S_A$ is the canonical projection.

Now, for any set, called an *alphabet*, we construct a free semi-group. Let Z be an arbitrary alphabet. By $F(Z)$ we denote the set of all words in the alphabet Z , i.e., $F(Z)$ is the set of all finite sequences of elements of Z . We assume that the *empty word* (i.e., the empty sequence), denoted by e , also belongs to $F(Z)$. For two words $w = (z_1, \dots, z_n)$ and $w' = (z'_1, \dots, z'_m)$, we define the *product*

$$ww' = (z_1, \dots, z_n, z'_1, \dots, z'_m).$$

This operation induces a semi-group structure on $F(Z)$. Of course, the empty word e has the property: $ew = we = w$, for each word w ($F(Z)$ satisfies the left cancellation law).

Let $g: Z \rightarrow Z'$ be any map of two alphabets. Then there exists a unique extension $\hat{g}: F(Z) \rightarrow F(Z')$ of g to a semi-group homomorphism. This extension is given by the formula

$$\hat{g}(z_1, \dots, z_n) = (g(z_1), \dots, g(z_n))$$

for each $w = (z_1, \dots, z_n)$ in $F(Z)$.

The above construction will be used for input and output alphabets of Mealy-Moore sequential machines. Let A be any Mealy-Moore machine. The functions δ_A and λ_A , defined on some subset of $S_A \times X_A$, may be extended to functions defined on some subset of $S_A \times F(X_A)$ as follows (we denote the extensions of δ_A and λ_A by the same symbols). δ_A is defined at $(a, w) \in S_A \times F(X_A)$, where $w = (x_1, \dots, x_n)$, if and only if the following symbols are defined

$$(*) \quad \delta_A(a, x_1) = a_1, \dots, \delta_A(a_{n-1}, x_n) = a_n,$$

and we put $\delta_A(a, w) = a_n$. Next, λ_A is defined at (a, w) if and only if δ_A is defined at (a, w) and all symbols

$$\lambda_A(a_1, x_1) = y_1, \dots, \lambda_A(a_n, x_n) = y_n,$$

are defined where a_1, \dots, a_n are defined by $(*)$ and we put

$$\lambda_A(a, w) = (y_1, \dots, y_n) \in F(Y_A).$$

Proposition 1. *The 5-tuple $A = (S_A, F(X_A), F(Y_A), \delta_A, \lambda_A)$ is a machine in the sense of Definition 1.*

For a lot of results formulated for machines [3], [8] or Mealy-Moore machines [2], [4], [5], the input semi-group is unimportant. For this reason, in many cases, it is sufficient to consider machines consisting only of a set of states, an output semi-group and a next state function (see [7], [4]). These machines are called by Gluszkow ([4], p. 41) *Miedviediev machines*. The precise definition is the following

Definition 5. A *Miedviediev machine* is a triple $A = (S_A, W_A, \delta_A)$ in which S_A , W_A and δ_A are as in Definition 1.

Let A be any sequential machine. Then $A' = (S_A, W_A, \delta_A)$ is Miedviediev machine. For $a \in S_A$ and $w \in W_A$ we denote $a \cdot w = \delta_A(a, w)$, whenever δ_A is defined at (a, w) . Now Definitions 1 and 5 immediately imply the following

Proposition 2. *A triple $ob(A) = ob(A') = (S_A, W_A, \cdot)$ has the following properties:*

- (1) W_A is a semi-group with the unit e ,
- (2) for each $a \in S_A$, the product $a \cdot e$ is defined and $a \cdot e = a$,
- (3) for each $a \in S_A$ and $w, w' \in W_A$, the products $a \cdot w$ and $(a \cdot w) \cdot w'$ are defined if and only if the product $a \cdot (ww')$ is defined, and $(a \cdot w) \cdot w' = a \cdot (ww')$.

Hence, $ob(A)$ is a right algebraic object over W_A (cf. Definition 4 in [10], p. 68; the above properties are a little different than in [10]).

Let A be any machine. We define an equivalence relation R on S_A as follows. For $w, w' \in W_A$, w and w' are equal modulo R if, for each $a \in S_A$, the product $a \cdot w$ is defined if and only if the product $a \cdot w'$ is defined, and, whenever these products are defined, $a \cdot w = a \cdot w'$. It is easy to see that the operation

$$[w][w'] = [ww']$$

induces (at least) a semi-group structure on the factorset W_A^* of W_A by R . Also the product $a \cdot [w] = a \cdot w$ is well defined. Thus we obtain

Proposition 3. *A triple $ob^*(A) = (S_A, W_A^*, \cdot)$ is a right algebraic object over W_A , i.e., $ob^*(A)$ satisfies the conditions (1)–(3) of Proposition 2.*

$ob^*(A)$ is called the *reduced algebraic object associate to a (Miedviediev) machine A* .

Thus, by Propositions 2 and 3, the theory of Miedviediev machines, from the algebraic point of view, is equivalent to the theory of (right) algebraic objects.

