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On Convex Solutions of the Functional Equation $\varphi^2(x) = g(x)$

Let the function $g: E \rightarrow R$, where $E \subset R$ is an interval, fulfil the following hypothesis:

(H) g is defined, convex and strictly increasing in an interval $[\alpha, \beta]$ and fulfils the condition $0 < g(x) < x$ in (α, β) .

For a function φ , we denote by φ^n its n -th iterate:

$$\begin{aligned} \varphi^0(x) &= x, \\ \varphi^{k+1}(x) &= \varphi(\varphi^k(x)), \quad \varphi^{-k-1}(x) = \varphi^{-1}(\varphi^{-k}(x)) \quad (k = 0, 1, \dots). \end{aligned}$$

We shall consider the functional equation

$$(1) \quad \varphi^2(x) = g(x).$$

A. Smajdor [2] has proved that if the function g is convex and strictly increasing in (α, β) , and fulfils hypothesis (H), and if g has one point of nondifferentiability in this interval, then there is no convex and increasing solution of equation (1).

In the present paper we consider an example of equation (1) in which the function g is increasing, convex and differentiable in a certain interval and which has no increasing and convex solution in this interval.

We start with some lemmas.

For a function $f: (\alpha, \beta) \rightarrow R$, the set of all points $x \in (\alpha, \beta)$ at which f is not differentiable is denoted by Z_f .

Lemma 1 ([2]). *If a function f fulfils hypothesis (H) then $Z_f \subset Z_{f^2}$.*

In [2] the conditions $f(\alpha) = \alpha$ and $f(\beta) = \beta$ are postulated, but they are not used in the proof of the above lemma.

Lemma 2. *If φ is an increasing and continuous solution of equation (1) in the interval (α, β) and if the function g fulfils hypothesis (H), then*

$$(2) \quad 0 < g(x) < \varphi(x) < x.$$

Proof. If, for an $x \in (\alpha, \beta)$, $\varphi(x) \geq x$, then $g(x) = \varphi(\varphi(x)) \geq \varphi(x) \geq x$ which contradicts (H). Similarly, if, for an $x \in (\alpha, \beta)$, $g(x) \geq \varphi(x)$, then $\varphi(g(x)) \geq \varphi^2(x)$

$= g(x)$, in contradiction with the inequality just proved. Thus for every $x \in (\alpha, \beta)$, we have (2).

Inequality (2) implies that

$$\lim_{x \rightarrow 0+} \varphi(x) = 0.$$

Theorem 1. *The unique convex and increasing solution of the equation*

$$(3) \quad \varphi^2(x) = sx \quad (0 < s < 1)$$

in the interval $[0, \beta)$ is given by the formula

$$(4) \quad \varphi(x) = \sqrt{sx}.$$

Proof. Let φ be an increasing and convex solution of equation (3). Since the function $g(x) = sx$ is differentiable in $[0, \beta)$, by Lemma 1 the function φ is differentiable. Its derivative is a nonnegative increasing function. Therefore

$$\lim_{x \rightarrow 0+} \varphi'(x)$$

exists and is finite. From equation (3) we obtain in $[0, \beta)$:

$$(5) \quad \varphi'(\varphi(x))\varphi'(x) = s.$$

According to Lemma 2 the function φ fulfils inequality $0 < sx < \varphi(x) < x$, whence

$$(6) \quad \lim_{x \rightarrow 0+} \varphi'(\varphi(x)) = \lim_{x \rightarrow 0+} \varphi'(x).$$

By (5), (6) and the fact that the function φ is continuous, we have

$$[\lim_{x \rightarrow 0+} \varphi'(x)]^2 = \lim_{x \rightarrow 0+} [\varphi'(\varphi(x))\varphi'(x)] = s.$$

Therefore

$$\lim_{x \rightarrow 0+} \varphi'(x) = \sqrt{s}.$$

Hence, for an arbitrary point x of the interval $[0, \beta)$, we have

$$(7) \quad \varphi'(x) \geq \sqrt{s}.$$

Suppose that there exists a point $x_0 \in [0, \beta)$ such that

$$(8) \quad \varphi'(x_0) > \sqrt{s}.$$

Inequality (7) implies that

$$\varphi'(\varphi(x_0)) \geq \sqrt{s}.$$

From this and from inequality (8) we get

$$s = \varphi'(\varphi(x_0))\varphi'(x_0) > \sqrt{s}\sqrt{s} = s,$$

which is impossible. Thus

$$\varphi'(x) = \sqrt{s} \quad x \in [0, \beta).$$

Hence and from inequality (2) we obtain

$$\varphi(x) = \sqrt{s}x \quad x \in [0, \beta).$$

Theorem 2. For the function g given by the formula

$$(9) \quad g(x) = \begin{cases} sx & \text{for } x \in (0, x_0] \\ ax + b & \text{for } x \in [x_0, \beta), \end{cases}$$

where $0 < s < 1$, $s < a$ and $ax_0 + b = sx_0$, $g(x) < x$ in $(0, \beta]$, equation (1) has no convex and increasing solution in $(0, \beta)$.

Proof. Suppose that there exists a convex and increasing solution φ of equation (1) in $(0, \beta)$. Then φ restricted to the interval $(0, x_0]$ is an increasing and convex solution of equation (1) in that interval. By Theorem 1, $\varphi(x) = \sqrt{s}x$ for $x \in (0, x_0]$. Let

$$(10) \quad x_n = \varphi^{-n}(x_0) \quad (n = 0, \pm 1, \dots).$$

We shall prove by induction that the function φ is linear in each interval $[x_n, x_{n+1}]$. From equation (1), we obtain:

$$(11) \quad \varphi(x) = \varphi^{-1}(g(x)).$$

Function φ is linear in the interval $[x_{-1}, x_0]$, so $\varphi^{-1}(x)$ is linear in the interval $[x_{-2}, x_{-1}]$. For $x \in [x_0, x_1]$, $g(x) \in [x_{-2}, x_{-1}]$, thus $\varphi(x) = \varphi^{-1}(g(x))$ is linear as the superposition of linear functions.

Suppose that φ is linear in an interval $[x_{n-1}, x_n]$. We shall prove that φ is linear in the interval $[x_n, x_{n+1}]$. The function φ fulfils condition (11). Let $x \in [x_n, x_{n+1}]$; then $g(x) \in [x_{n-2}, x_{n-1}]$ and $\varphi(x) \in [x_{n-1}, x_n]$. The function φ is linear in the interval $[x_n, x_{n+1}]$, as superposition of linear functions, since the function φ is linear in the interval $[x_{n-1}, x_n]$. Therefore there exist numbers a_n, b_n such that

$$\varphi(x) = a_n x + b_n \quad x \in [x_{n-1}, x_n].$$

Condition (11) implies that $\varphi^{-1}(x) = g^{-1}(\varphi(x))$. Putting $g(x)$ instead of x , we have

$$(12) \quad \varphi(x) = \varphi^{-1}(g(x)) = g^{-1}(\varphi(g(x))).$$

The direction coefficient of the function $\varphi(x) = g^{-1}(\varphi(g(x)))$ is equal to the product of direction coefficients of the respective functions. Thus the direction coefficient of the function $g^{-1}(\varphi(g(x)))$ is equal to the direction coefficient of the function φ .

Take $x \in [x_{n-1}, x_n]$. Then $\varphi(x) = a_n x + b_n$, and $g(x) \in [x_{n-3}, x_{n-2}]$, the direction coefficient of the function $g^{-1}(\varphi(g(x)))$ equals a_{n-2} . Therefore, for an arbitrary $n \geq 0$, we have

$$(13) \quad a_n = a_{n-2}.$$

But if $x \in [x_0, x_1]$, then

$$\varphi(x) = \varphi^{-1}(g(x)) = \frac{ax+b}{\sqrt{s}} = \frac{a}{\sqrt{s}}x + \frac{b}{\sqrt{s}} = a_1x + b_1.$$

Thus $a_1 \neq a_0$ and from (13) we obtain that the function φ cannot be convex in the interval $(0, \beta)$.

The above argument yields simultaneously a proof of the following

Theorem 3. *If φ is a continuous and increasing solution of equation (1) in the interval $(0, \beta)$, convex in the interval $(0, x_0]$, and if function g is given by formula (9) then the set Z_φ is infinite.*

Theorem 3 follows from Smajdor's result [2], but the above proof is different. Let the function g be given by the formula

$$(14) \quad g(x) = \begin{cases} sx & \text{for } x \in (0, x_0] \\ h(x) & \text{for } x \in [x_0, x_1] \\ ax+b & \text{for } x \in [x_1, \beta). \end{cases}$$

where $x_1 > x_0$ is an arbitrary point of the interval $(0, \beta)$, $h(x_1) = \sqrt{s}x_0$ and $0 < s < 1$, $s < a$.

Take the function h so that the function g be differentiable in the interval $(0, \beta)$ and fulfil hypothesis (H) in this interval.

Lemma 3. *If for g given by formula (14), φ is a convex and increasing solution of equation (1), then for every $n \geq 0$ we have*

$$\varphi'(x_{n+2}) = \varphi'(x_n).$$

Proof. The function g being differentiable in the interval $(0, \beta)$, by Lemma 1 the function φ is also differentiable in that interval. Differentiating equation (1) we have

$$\varphi'(\varphi(x))\varphi'(x) = g'(x),$$

whence

$$(15) \quad \varphi'(x) = \frac{g'(x)}{\varphi'(\varphi(x))}.$$

Putting $\varphi(x)$ instead of x we obtain

$$\varphi'(\varphi(x)) = \frac{g'(\varphi(x))}{\varphi'(\varphi^2(x))} = \frac{g'(\varphi(x))}{\varphi'(g(x))}.$$

Inserting the expression obtained to formula (15) we obtain

$$\varphi'(x) = \frac{g'(x)}{g'(\varphi(x))}\varphi'(g(x)).$$

Setting $x = x_{n+2}$ we get

$$\varphi'(x_{n+2}) = \frac{g'(x_{n+2})}{g'(\varphi(x_{n+2}))} \varphi'(g(x_{n+2})).$$

In view of (10), $g(x_{n+2}) = x_n$ and $\varphi(x_{n+2}) = x_{n+1}$. Hence we obtain from the above relation

$$\varphi'(x_{n+2}) = \frac{g'(x_{n+2})}{g'(x_{n+1})} \varphi'(x_n).$$

But for $x \geq x_1$ we have $g'(x) = a$, which completes the proof.

Theorem 4. *For the function g given by formula (14), equation (1) has no convex and increasing solution in the interval $(0, \beta)$.*

Proof. Let φ be a convex and increasing solution of the equation under consideration.

Putting in formula (15) $x = x_1$ we obtain

$$\varphi'(x_1) = \frac{g'(x_1)}{\varphi'(\varphi(x_1))} = \frac{g'(x_1)}{\varphi'(x_0)} = \frac{a}{\sqrt{s}},$$

since, by Theorem 2, $\varphi(x) = \sqrt{s}x$ in the interval $(0, x_0]$. By hypothesis, $a \neq s$, whence $\frac{a}{\sqrt{s}} \neq \sqrt{s}$. This implies that

$$\varphi'(x_1) \neq \sqrt{s} = \varphi'(x_0) = \varphi'(x_2).$$

The assertion of Theorem 4 results now directly from Lemma 3.

REFERENCES

- [1] M. Kuczma, *On the functional equation $\varphi^n(x) = g(x)$* , Ann. Polon. Math. 11 (1961), 161—165.
 [2] A. Smajdor, *On superposition of convex functions*, Arch. Math. 17 (1966), 333—335.