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On the Rapidity of Convergence of Sequences of Errors in the Methods of Newton and of Regula Falsi

It has been proved by S. Gołąb [1] that the rapidity of the convergence to zero of the sequence $\{\sigma_n\}$ of errors of approximation in the method of Newton is greater than that of the sequence $\{\varrho_n\}$ of errors of approximation in the method of regula falsi. Namely,

$$\lim_{n\to\infty}\frac{\sigma_n}{\varrho_n}=0 \tag{1}$$

under the assumption that the left-hand side of the equation

$$f(x)=0 \tag{2}$$

is a function of class C^2 and its first and second derivatives have constant sign in an interval $[a,b]$ at the ends of which the function $f(x)$ has different signs. S. Gołąb has suggested to investigate the same problem under weaker assumptions on $f(x)$.

In the present paper we shall show that (1) is satisfied (with a suitable generalization of Newton's method) under the assumption that $f(x)$ is strictly monotone and strictly convex in the interval $[a,b]$ and that $f(a)\cdot f(b)<0$.

Without loss of generality we may assume that $f(x)$ is strictly increasing.

It is easily seen that the function $f(x)$ is continuous in $[a,b]$ and at every point x of that interval has finite left-hand and right-hand derivatives, $f'_-(x)$ and $f'_+(x)$. Moreover, for every $x_1, x_2 \epsilon (a,b)$, $x_1<x_2$ implies that

$$0<f'_-(x_1)\geqslant f'_+(x_1)\leqslant f'_-(x_2)\leqslant f'_+(x_2)\,. \tag{3}$$

We also easily see that the algorithm of the method regula falsi determines in our case a strictly increasing and bounded from above sequence $\{x_n\}$ of approximations of the unique root of equation (2),

$$x_1=\frac{f(b)a-f(a)b}{f(b)-f(a)}\,,\qquad x_{n+1}=\frac{f(b)-bf(x_n)}{f(b)-f(x_n)}\qquad (n=1,2,\ldots) \tag{4}$$

since from $x_n \to x_0$ it follows, by the continuity of $f(x)$, that

$$x_0 = \frac{f(b)x_0 - bf(x_0)}{f(b)-f(x_0)}$$

and hence $f(x_0) = 0$.

We can also easily prove that the algorithm of the method of Newton (with $f'(x)$ replaced by $f'_+(x)$) determines a decreasing and bounded from below sequence $\{z_n\}$,

$$z_1 = b - \frac{f(b)}{f'_-(b)}, \quad z_{n+1} = z_n - \frac{f(z_n)}{f'_+(z_n)} \quad (n = 1, 2, \ldots) \tag{5}$$

which converges to the root x_0 since, by the continuity of $f(x)$ and the boundedness of the sequence $\{f'_+(z_n)\}$, from $z_n \to z_0$ it follows that

$$z_0 = z_0 - \frac{f(z_0)}{\lim\limits_{n\to\infty} f'_+(z_n)},$$

so that $f(z_0) = 0$ and therefore $z_0 = x_0$.

In the sequel we will need the following lemma whose proof, quite similar to that of the standard Lagrange theorem, will be left to the reader.

Lemma. *If a function $f(x)$ is continuous in an interval $[a, b]$ and has there left-hand and right-hand derivatives (finite or not), then there exists a $\xi \in (a, b)$ such that*

$$f'_-(\xi) \leqslant \frac{f(b)-f(a)}{b-a} \leqslant f'_+(\xi).$$

From (4) and (5) we get

$$z_n - z_{n+1} = \frac{f(z_n)}{f'_+(z_n)}, \quad x_{n+1} - x_n = \frac{f(x_n)(x_n - b)}{f(b)-f(x_n)}.$$

Hence

$$\varrho_{n+1} = x_0 - x_{n+1} + x_n - x_n = \varrho_n - \frac{f(x_n)(x_n-b)}{f(b)-f(x_n)}, \tag{6}$$

$$\sigma_{n+1} = z_{n+1} - z_n + z_n - x_0 = \sigma_n - \frac{f(z_n)}{f'_+(z_n)}. \tag{7}$$

Applying the lemma to the function $f(x)$ in the intervals $[x_n, x_0]$, $[x_0, z_n]$, $[x_n, b]$ we get

$$f(x_n) = f(x_n) - f(x_0) = -\varrho_n \alpha_n, \quad f'_-(\xi_n) \leqslant \alpha_n \leqslant f'_+(\xi_n), \tag{8}$$

$$f(z_n) = f(z_n) - f(x_0) = \sigma_n \beta_n, \quad f'_-(\zeta_n) \leqslant \beta_n \leqslant f'_+(\zeta_n), \tag{9}$$

$$f(b) - f(x_n) = (b - x_n)\gamma_n, \quad f'_-(\eta_n) \leqslant \gamma_n \leqslant f'_+(\eta_n) \tag{10}$$

with

$$x_n < \xi_n < x_0 < \zeta_n < z_n, \quad x_n < \eta_n < b.$$

From (6), (8) and (7), (9) it follows that

$$(11) \qquad \varrho_{n+1}=\varrho_n-\frac{\varrho_n\alpha_n}{\gamma_n}=\varrho_n\left(1-\frac{\alpha_n}{\gamma_n}\right),$$

$$(12) \qquad \sigma_{n+1}=\sigma_n-\frac{\sigma_n\beta_n}{f'_+(z_n)}=\sigma_n\left(1-\frac{\beta_n}{f'_+(z_n)}\right).$$

The sequence $\{\gamma_n\}$ tends, by (10), to $\gamma_0=f(b)/(b-x_0)$, where $x_0<\eta_0<b$. Because σ_n and ϱ_n are positive (sequences $\{x_n\}$, $\{z_n\}$ are strictly monotone), we can divide (12) by (11),

$$(13) \qquad \frac{\sigma_{n+1}}{\varrho_{n+1}}=\frac{\sigma_n\left(1-\dfrac{\beta_n}{f'_+(z_n)}\right)}{\varrho_n\left(1-\dfrac{\alpha_n}{\gamma_n}\right)}.$$

The sequences $\{f'_+(z_n)\}$, $\{\beta_n\}$, as decreasing and bounded from below by $f'_+(x_0)$, converge to finite limits. Likewise the sequence $\{\alpha_n\}$, as increasing and bounded from above by $f'_-(x_0)$, has a finite limit. From (3) we get the inequalities

$$(14) \qquad f'_-(\xi_n)\leqslant\alpha_n\leqslant f'_+(\xi_n)\leqslant f'_-(\zeta_n)\leqslant\beta_n\leqslant f'_+(\zeta_n)\leqslant f'_+(z_n)$$

and, for n greater than a certain N, the inequality

$$(15) \qquad f'_+(z_n)<\gamma_0.$$

By (14) and (15) we have

$$\lim_{n\to\infty}\beta_n\geqslant\lim_{n\to\infty}\alpha_n, \qquad \lim_{n\to\infty}f'_+(z_n)<\gamma_0.$$

Hence

$$\frac{\lim\limits_{n\to\infty}\beta_n}{\lim\limits_{n\to\infty}f'_+(z_n)}>\frac{\lim\limits_{n\to\infty}\alpha_n}{\gamma_0}.$$

Now, from (13), (14) and (15) it follows that

$$0\leqslant\lim_{n\to\infty}\frac{\dfrac{\sigma_{n+1}}{\varrho_{n+1}}}{\dfrac{\sigma_n}{\varrho_n}}<1,$$

and this means that (1) holds true.

REFERENCE

[1] S. Gołąb, *La comparaison de la rapidité de convergence des approximations successives de la méthode de Newton avec la méthode de „regula falsi“*, Mathematica 8 (1966), 45—49.