ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

CCLII PRACE MATEMATYCZNE, ZESZYT 15 1971

Stanisław Białas

Estimation of Solutions and Their Derivatives for Linear Differential Equations with Constant Coefffcients

1. The estimation accuracy of approximate solutions of differential equations given by analog computers depends on the possibility of an a priori estimation of exact solutions and their derivatives. In this note such estimations for n-th order linear differential equations with constant coefficients are given.

This paper presents a generalization of the results of [1].

2. Consider the n-th order differential equation

$$y^{(n)}+a_{n-1}y^{(n-1)}+\ldots+a_1 y+a_0 y=0 \tag{1}$$

with constant, real coefficients. Denote by $f(p)$ the characteristic polynomial of (1) and let p_i be the roots of $f(p)$. Denote by $y(t)$ the solution of (1) satisfying the initial conditions

$$y^{(i)}(0)=A_1 \quad (i=0\ldots n-1). \tag{2}$$

It is known that if $Re(p_i)<0$ $(i=1\ldots n)$, then the functions $y^{(k)}(t)$ $(k=0,1\ldots)$ are bounded for $t\geqslant 0$. In [2] the following problem is posed: knowing the coefficients of equation (1) and the initial values A_i, determine the estimations of $y^{(k)}(t)$ $(k=1,\ldots,n)$. These majorants should be obtained without knowing the characteristic roots of the equation.

Assume that the roots of $f(p)$ satisfy $p_i\neq p_j$ for $i\neq j$ $(i,j=1,\ldots,n)$ and introduce the following notations

$$\Delta_i=f'(p_i) \quad (i=1,\ldots,n) \tag{3}$$

$$\varepsilon=-\sum_{i=1}^{n}\frac{e^{p_i t}}{p_i^n\Delta_i}, \quad \varepsilon^{(k)}=\frac{d^k\varepsilon}{dt^k}, \quad \delta_i=\sum_{k=0}^{n-i-1}a_k A_{k+i} \quad (i=0,\ldots,n). \tag{4}$$

It is easy to verify that

$$y(t)=\sum_{i=0}^{n-1}\varepsilon^{(n-i-1)}\cdot\delta_i. \tag{5}$$

Let us consider moreover the polynomials

$$(6)\qquad f_j(p)=\prod_{\substack{i=1\\ i\neq j}}^{n}(p-p_j)\qquad (j=1,\dots,n).$$

Let us consider now the linear differential equations with constant coefficients

$$(7)\qquad x^{(n-1)}+b_{n-2,j}x^{(n-2)}+\dots+b_{0,j}x=0\qquad (j=1,\dots,n)$$

whose characteristic polynomials are (6).
Define

$$\Delta_{i,j}=f_j'(p_i)\qquad (i,j=1,\dots,n;\,i\neq j)$$

$$(8)\qquad \varepsilon_j=-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{e^{p_it}}{p_i^n\Delta_{i,j}},\qquad \varepsilon_j^{(k)}=\frac{d^k\varepsilon_j}{dt^k}\qquad (j=1,\dots,n)$$

$$\delta_{i,j}=\sum_{k=0}^{n-i-2}b_{k,j}\cdot A_{k+i}\qquad (i=0,\dots,n;\,j=1,\dots,n).$$

Let $x_j(t)$, $z_j(t)$ $(j=1,\dots,n)$ be the solutions of (7) satysfying the initial conditions:

$$x_j^{(k)}(0)=A_k,\qquad z_j^{(k)}(0)=A_{k+1}\qquad (k=0,\dots,n-2;\,j=1,\dots,n).$$

3. We shall prove that

$$(9)\qquad \varepsilon^{(k+1)}=p_j\varepsilon^{(k)}+\varepsilon_j^{(k)}$$

$$(10)\qquad y^{(k+1)}(t)=p_j\big(y^{(k)}(t)-x_j^{(k)}(t)\big)+z_j^{(k)}(t)\qquad (j=1,\dots,n)$$

for arbitrary k.
We present the proof for $k=0$. From (4), (8) we have

$$p_j\varepsilon+\varepsilon_j=-p_j\sum_{i=1}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{e^{p_it}}{p_i^n\Delta_{i,j}}=-p_i\sum_{i=1}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{(p_i-p_j)e^{p_it}}{p_i^n(p_i-p_j)\Delta_{i,j}}$$

$$=-p_j\sum_{i=1}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}+p_j\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{e^{p_it}}{p_i^{n-1}\Delta_i}$$

$$=-p_j\sum_{i=1}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}+p_j\sum_{i=1}^{n}\frac{e^{p_it}}{p_i^n\Delta_i}-\frac{p_je^{p_jt}}{p_j^n\Delta_j}-\sum_{\substack{i=1\\ i\neq j}}^{n}\frac{e^{p_it}}{p_i^{n-1}\Delta_i}=\varepsilon'.$$

From (5), (8) and the definition of the functions $x_j(t)$, $z_j(t)$ we have

$$p_j[y(t)-x_j(t)]+z_j(t)=p_j\Big[\sum_{i=0}^{n-1}\varepsilon^{(n-i-1)}\delta_i-\sum_{i=0}^{n-2}\varepsilon_j^{(n-i-1)}\delta_{i,j}\Big]+\sum_{i=0}^{n-2}\varepsilon_j^{(n-i-1)}\Big[\sum_{k=0}^{n-i-2}b_{k,j}A_{k+i+1}\Big]$$

$$=\sum_{i=0}^{n-1}p_j\varepsilon^{(n-i-1)}\delta_i+\sum_{i=0}^{n-2}\varepsilon_j^{(n-i-1)}\Big[\delta_i+A_{n-1}p_j\sum_{i=0}^{n-2}\varepsilon_j^{(n-i-1)}b_{n-1-i,j}\Big]$$

$$=\sum_{i=0}^{n-2}\delta_i\,[p_j\varepsilon^{(n-i-1)}+\varepsilon_j^{(n-i-1)}]+p_j\delta_{n-1}\varepsilon+A_{n-1}p_j\sum_{i=0}^{n-2}\varepsilon_j^{(n-i-1)}b_{n-i-1,j}$$

$$=\sum_{i=0}^{n-2}\delta_i\varepsilon^{(n-i)}+p_j\delta_{n-1}\varepsilon+p_jA_{n-1}[-b_{0,j}\varepsilon_j]$$

$$=\sum_{i=0}^{n-2}\delta_i\varepsilon^{(n-i)}+\delta_{n-1}\varepsilon'=y'(t)\,.$$

These relations were obtained with the assumption that $p_i\neq p_j$ $(i\neq j)$ but they are also valid for multiple roots of the characteristic polynomial. This can proved by examining the limit of the left and right members of equality (10) as $p_i\to p_j$.

4. Theorem. *If all roots of the characteristic polynomial of* (1) *are real and negative, then for* $t\geqslant 0$

$$|a_0\,y(t)|\leqslant\sum_{i=0}^{n-1}a_i|A_i|\,\frac{n-i}{n},$$

$$|a_k y^{(k)}(t)|\leqslant\frac{2^{k-1}}{n}\binom{n}{k}\Big[\sum_{i=0}^{n-1}(n-i)a_i|A_i|\Big]\qquad(k=1,\dots,n)\,.$$

Proof. Using (10) it is easy to verify that the assertion of Theorem is true for $n=2$. For the inductive proof assume that the Theorem is satisfied for equations of order $n-1$. Thus we have

(11)
$$|b_{0,j}x_j(t)|\leqslant\sum_{i=0}^{n-2}b_{i,j}|A_i|\,\frac{n-1-i}{n-1}$$

$$|b_{k,j}x_j^{(k)}(t)|\leqslant\frac{2^{k-1}}{n-1}\binom{n-1}{k}\Big[\sum_{i=0}^{n-2}(n-1-i)b_{i,j}|A_i|\Big]$$

$(j=1,\dots,n;\ k=1,\dots,n-1)$.

From (5) it follows that

$$\lim_{t\to\infty}y^{(k)}(t)=0\qquad(k=0,1\,\dots)\,.$$

Therefore for each $k=0,1,\ldots$ the functions $y^{(k)}(t)$ are bounded in $[0,\infty)$. If $y^{(k)}(0)\geqslant y^{(k)}(t)$. Theorem is satisfied automatically. So assume that $y^{(k)}(t)$ takes its maximum for $t_k>0$. We have $y^{(k+1)}(t_k)=0$, hence from (10) it follows that

$$-p_j y^{(k)}(t_k) = -p_j x_j^{(k)}(t_k)+z_j^{(k)}(t_k) \quad (j=1,\ldots,n).$$

Multiply each of these equalities respectively by $b_{k,j}$ and sum. We obtain

$$-\sum_{j=1}^{n} b_{k,j}p_j y^{(k)}(t_k) = -\sum_{j=1}^{n} b_{k,j}p_j x_j^{(k)}(t_k)+\sum_{j=1}^{n} b_{k,j}z_j^{(k)}(t_k)$$

or, since

$$-\sum_{j=1}^{n} b_{k,j}p_j = (n-k)a_k,$$

(12)
$$(n-k)a_k y^{(k)}(t_k) = \sum_{j=1}^{n} b_{k,j}[-p_j x_j^{k}(t_k)+z_j^{(k)}(t_k)].$$

We shall consider the cases $k=0$, $0<k<n$, $k=n$ separately. In the case $k=n$ from (12) and (11) it results

$$na_0|y(t)| \leqslant -\sum_{j=1}^{n} p_j\Big[\sum_{i=0}^{n-2} b_{i,j}|A_i|\frac{n-1-i}{n-1}\Big]+\sum_{j=1}^{n}\Big[\sum_{i=0}^{n-2} b_{i,j}|A_{i+1}|\,\frac{n-1-i}{n-1}\Big]$$

$$= na_0|A_0|+\frac{1}{n-1}\Big[\sum_{i=0}^{n-3}|A_{i+1}|\Big(\sum_{j=1}^{n}\big(b_{i,j}(n-1-i)-p_j b_{i+1,j}(n-2-i)\big)\Big)\Big]+a_{n-1}|A_{n-1}|$$

$$= na_0|A_0|+\frac{1}{n-1}\Big[\sum_{i=1}^{n-2}|A_i|\Big(\sum_{j=1}^{n}\big(b_{i-1,j}(n-i)-p_j b_{i,j}(n-1-i)\big)\Big)\Big]+a_{n-1}|A_{n-1}|.$$

But

(13)
$$\sum_{j=1}^{n}[-p_j b_{i,j}(n-1-i)+b_{i-1,j}(n-i)] = (n-1)(n-i)a_i,$$

so

$$|a_0 y(t)| \leqslant \sum_{i=0}^{n-1} a_i|A_i|\frac{n-i}{n},$$

which proves the first part of the thesis of the Theorem. In the similar manner using (12) and (11) we obtain the estimations for $a_k y^{(k)}(t)$:

$$(n-k)|a_k y^{(k)}(t)| \leqslant -\sum_{j=1}^{n} p_j\Big[\frac{2^{k-1}}{n-1}\binom{n-1}{k}\Big(\sum_{i=0}^{n-2}(n-1-i)b_{i,j}|A_i|\Big)\Big]$$
$$+\sum_{j=1}^{n}\frac{2^{k-1}}{n-1}\binom{n-1}{k}\Big(\sum_{i=0}^{n-2}(n-1-i)b_{i,j}|A_{i+1}|\Big)$$

$$= \frac{2^{k-1}}{n-1}\binom{n-1}{k}\Big[(n-1)na_0A_0+$$

$$+\sum_{i=1}^{n-2}|A_i|\Big(\sum_{j=1}^{n}\big(b_{i-1,j}(n-i)-p_jb_{i,j}(n-1-i)\big)\Big)+(n-1)a_{n-1}|A_{n-1}|\Big].$$

which with (13) gives

$$(14)\qquad |a_k y^{(k)}(t)| \leqslant \frac{2^{k-1}}{n-k}\binom{n-1}{k}\Big[\sum_{i=0}^{n-1}(n-i)a_i|A_i|\Big] = \frac{2^{k-1}}{n}\binom{n}{k}\Big[\sum_{i=0}^{n-1}(n-i)a_i|A_i|\Big].$$

For $k = n$ from (10) we have

$$y^{(n)}(t) = p_j\big(y^{(n-1)}(t)-x_j^{(n-1)}(t)\big)+z_j^{(n-1)}(t) \qquad (j=1,\dots,n).$$

Summing up these equalities and applying (10), (11) we obtain

$$|ny^{(n)}(t)| = |-a_{n-1}y^{(n-1)}(t)-\sum_{j=1}^{n}p_jx_j^{(n-1)}(t)+\sum_{j=1}^{n}z_j^{(n-1)}(t)|$$

$$\leqslant \frac{2^{n-2}}{n}\binom{n}{n-1}\Big[\sum_{j=0}^{n-1}(n-j)a_j|A_j|\Big]+\frac{2^{n-2}}{n-1}(n-1)\Big[\sum_{j=0}^{n-1}(n-j)a_j|A_j|\Big]$$

$$= 2^{n-1}\Big[\sum_{j=0}^{n-1}(n-j)a_j|A_j|\Big],$$

which completes the proof.

REFERENCES

[1] **S. Białas**, *Oszacowanie rozwiązań i ich pochodnych równań różniczkowych liniowych*, Archiwum Automatyki i Telemechaniki 12 (1967), 124—131.

[2] **A. S. Jacson**, *Analog Computation*, New York 1960.