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On the Existence of Solutions of Linear Boundary-value Problem for Ordinary Differential Equations

This note concerns the existence and uniqueness of solutions of linear boundary value problems for systems of first-order non-linear ordinary differential equations. It presents the generalization of our earlier note [3] and gives a different proof of a certain result due to A. Lasota and Z. Opial [8]. In contrast to [8], we do not use the theory of contingent equations (Lemma of Pliś [9]) but we base on the Leray-Schauder fixed-point theorem.

Let R^n denote the n-dimensional Euclidean space, with the Euclidean norm $|x|$. Let $C_{[0,h]}$ ($C^{\infty}_{[0,h]}$) be the space of all continuous (C^{∞}) mappings of $[0, h]$ into R^n with the norm of uniform convergence, $||x|| = \max\{|x(t)| : t \in [0, h]\}$.

We shall deal with the systems of ordinary differential equations

$$x' = f(t, x) \tag{1}$$

and a boundary-value condition

$$Lx = r, \tag{2}$$

where $f: [0, h] \times R^n \to R^n$ and the mapping $L: C_{[0,h]} \to R^n$ is linear and continuous.

Moreover we shall consider the differential inequalities of the form

$$|x'| \leqslant p(t)|x| + q(t), \tag{3}$$

where the functions $p(t)$ and $q(t)$ are nonnegative and summable.

By the *solution* of (1) (resp. (3)) we mean any absolutely continuous function $x(t) \in C_{[0,h]}$ satisfying (1) (resp. (3)) almost everywhere on $[0, h]$.

We shall assume in the sequel that:

(C) For each fixed $t \in [0, h]$, $f(t, x)$ is continuous in x, for each fixed $x \in R^n$, $f(t, x)$ is Lebesgue measurable with respect to t.

(L) $x(t) \equiv 0$ is the unique solution of the equation $x' = 0$ satisfying $Lx = 0$.

It may be proved (the proof follows easily from [1]) that (L) is equivalent to the following condition:

($L1$) The restriction L_k of L to the subspace of $C_{[0,h]}$ consisting of all constant functions is invertible.

In section 1 we consider certain properties of differential inequalities associated with the boundary value problem (1), (2). In section 2 we prove a lemma which is used to obtain an a priori estimate of solutions of (1), (2). The existence theorems are presented in section 3. In section 4 we show that for to verify the assumptions of the existence theorem it is enough to prove that a certain differential inequality has no nontrivial solutions of class C^∞ (Theorem 4). In the last section the application of obtained theorems to the Floquet's problem is discussed.

1. Assume that L is linear, continuous and satisfies (L) and consider the differential inequality

$$|x'| \leqslant p(t)|x| \tag{4}$$

with the homogeneous boundary value condition

$$Lx = 0\,. \tag{5}$$

Theorem 1. *Under above assumptions, there is a positive constant M such that if $p(t)$ satisfies*

$$\int_0^h p(t)\,dt < M\,, \tag{6}$$

then problem (4), (5) *has only the trivial* (*null*) *solution.*

Theorem 1 is a consequence of a more general result of A. Lasota ([5], Theorem 2). For the convenience of the reader, however, we give here a simple direct proof of Theorem 1.

Proof. For an absolutely continuous function $x(t)$ satisfying (5) we have

$$x(t) = \int_0^t x'(s)\,ds + C \quad \text{and} \quad L\int_0^t x'(s)\,ds + LC = 0\,.$$

Since L satisfies ($L1$), L_k^{-1} exists. Hence

$$C = -L_k^{-1}L\int_0^t x'(s)\,ds \quad \text{and} \quad x(t) = (I - L_k^{-1}L)\int_0^t x'(s)\,ds$$

(I stands for the identity mapping).

Put $A = I - L_k^{-1}L$. Now, if $x(t)$ satisfies (4), then

$$||x|| \leqslant ||A||\cdot\left\|\int_0^t x'(s)\,ds\right\| \leqslant ||A||\cdot||x||\cdot\int_0^h p(s)\,ds\,,$$

where $||A||$ denotes the norm of A. Hence $M = ||A||^{-1}$ satisfies the assertion of Theorem 1.

2. Denote by M_L the least upper bound of the set of numbers M for which Theorem 1 holds. M_L is the "best estimate" for problem (5), (4). For some boundary value problems M_L has been found. For example, for initial-value problem

$(Lx = x(t_0)) - M_L = \infty$ for the Nicoletti's problem $(Lx = x_1(t_1), x_2(t_2), \ldots, x_n(t_n))$, ($x_i$ denotes the i-th component of x, $t_i \in [0, h]) - M_L = \pi/2$ [7], for the Floquet's problem $(Lx = x(0) + \lambda x(h),\ \lambda > 0) - M_L = \sqrt{\pi^2 + \ln^2\lambda}$ [3], for the problem $(Lx = x(0) + \lambda x(h),\ \lambda < 0) - M_L = |\ln|\lambda||$ [5].

Lemma 1. *Let $p(t)$ and $q(t)$ be summable and nonnegative on* $[0, h]$, *and let*

(7) $$\int_0^h p(t)\,dt < M_L\,.$$

Then every solution $x(t)$ of problem (3), (5) *satisfies the inequality*

(8) $$|x(t)| \leqslant \int_0^h q(t)\,dt\Big[1 + \Big(M_L - \int_0^h p(t)\,dt\Big)^{-1}\Big]\exp\Big(\int_0^h p(t)\,dt\Big)\,.$$

Proof. If $m = \min\{|x(t)| : t \in [0, h]\}$ and $m \neq 0$, then (3) implies that

$$|x'(t)| \leqslant \big(p(t) + q(t)/m\big)|x(t)|\,.$$

Hence, by the definition of M_L

$$\int_0^h \big(p(t) + q(t)/m\big)\,dt \geqslant M_L,$$

and therefore, by (7),

(9) $$m \leqslant \Big(M_L - \int_0^h p(t)\,dt\Big)^{-1}\int_0^h q(t)\,dt\,.$$

Obviously (9) remains valid if $m = 0$.

For the function $v(t) = |x(t)|$, the inequalities $|v'(t)| \leqslant |x'(t)|$ and (3) imply that

$$-p(t)\cdot v - q(t) \leqslant v' \leqslant p(t)\cdot v + q(t)\,.$$

Hence ([2], p. 27), for any $t_0 \in [0, h]$,

$$v(t) \leqslant \Big(v(t_0) + \int_{t_0}^t q(u)\exp\Big(-\int_{t_0}^u p(s)\,ds\Big)\,du\Big)\exp\Big(\int_{t_0}^t p(s)\,ds\Big) \quad \text{for} \quad t \in [t_0, h]\,,$$

$$v(t) \leqslant \Big(v(t_0) - \int_{t_0}^t q(u)\exp\Big(\int_{t_0}^u p(s)\,ds\Big)\,du\Big)\exp\Big(-\int_{t_0}^t p(s)\,ds\Big) \quad \text{for} \quad t \in [0, t_0].$$

Take t_0 such that $v(t_0) = m$. Then (9), the above inequalities and the inequalities

$$\int_0^t p(s)\,ds \leqslant \int_0^h p(s)\,ds\,, \quad \exp\Big(-\int_{t_0}^t p(s)\,ds\Big) \leqslant 1$$

imply (8).

3. Theorem 2. *Assume* (C) *and* (L). *Let $f(t, x)$ satisfy*

(10) $$|f(t, x)| \leqslant p(t)|x| + q(t) \quad \textit{for} \quad (t, x) \in [0, h] \times R^n,$$

where $p(t)$ and $q(t)$ are nonnegative and summable.

If $p(t)$ satisfies (7), *then problem* (1), (2) *has at least one solution.*

Proof. The change of variable $x = y - L_k^{-1}r$ reduces (1), (2) to the homogeneous boundary value problem (1), (5), hence, without loss of generality, we may consider problem (1), (5).

For $\mu \in [0, 1]$ define a family of mappings $T_\mu: C_{[0,h]} \to C_{[0,h]}$ by

$$T_\mu x(t) = x(t) - \mu \int_0^t f(s, x(s))ds - x(0) + Lx.$$

It is easy to verify that $T_\mu x = 0$ if and only if $x = x(t)$ is the solution of the equation

$$x' = \mu f(t, x)$$

satisfying (5). T_μ is linear, continuous in (μ, x) and completely continuous.

Let K be the ball of $C_{[0,h]}$, $K = \{x: \|x\| \leqslant A+1\}$, where A denotes the right-hand side of (8) and let S be the boundary of K.

Since the mapping T_0 is linear and continuous on S, its degree is equal to 1 ([4], p. 110). By (8), $T_\mu x \neq 0$ for $(\mu, x) \in [0, 1] \times S$. Hence the degrees of T_0 and T_μ $(\mu \in (0, 1])$ are equal ([4], p. 114). From the Leray-Schauder theory it follows that there is $x_0 \in K$ such that $T_1 x_0 = 0$, which completes the proof.

Theorem 3. *Assume* (C) *and* (L). *Let $f(t, x)$ satisfy*

(11) $\qquad |f(t, u) - f(t, v)| \leqslant p(t)|u - v| \quad$ *for* $\quad (t, u), (t, v) \in [0, h] \times R^n$

and let $p(t)$ be summable and satisfy (7).

Then problem (1), (2) *has exactly one solution.*

Proof. Since (11) implies (10), the existence of the solution of (1), (2) follows from Theorem 2. To prove the uniqueness, consider two solutions $x_1(t), x_2(t)$ of (1), (2). By (11), $x(t) = x_1(t) - x_2(t)$ is a solution of problem (4), (5). Since $p(t)$ satisfies (7), $x(t) \equiv 0$.

4. Lemma 2. *Assume* (L). *Then for any absolutely continuous function $x: [0, h] \to R^n$ satisfying* (5) *there is a sequence $\{x_n(t)\} \subset C^\infty_{[0,h]}$ such that*

$$\lim_{n\to\infty} \|x_n - x\| = 0, \quad Lx_n = 0.$$

Proof. Since $x'(t)$ is summable, there is a sequence $\{u_n(t)\} \subset C^\infty_{[0,h]}$ such that l.i.m. $u_n(t) = x'(t)$, (l.i.m. denotes mean convergence).

Put $x_n(t) = y_n(t) + c_n$, $x(t) = y(t) + x(0)$, where

$$y_n(t) = \int_0^t u_n(s)ds, \quad x(t) = \int_0^t x'(s)ds, \quad c_n = -L_k^{-1}Ly_n, \quad x(0) = -L_k^{-1}Ly.$$

It is easily seen that $\|y_n - y\| \to 0$ as $n \to \infty$. Hence, by the continuity of $L_k^{-1}L$,

$$\lim_{n\to\infty} (c_n - x(0)) = \lim_{n\to\infty} L_k^{-1}L(y - y_n) = 0.$$

This completes the proof of Lemma 2.

Let M_L^∞ be the least upper bound of numbers M such that the problem (4), (5) has only trivial solution of class C^∞ for all non-negative functions p of class C^∞ satisfying (6).

Theorem 4. *If L is linear, continuous and satisfies (L), then $M_L^\infty = M_L$.*

Proof. The inequality $M_L^\infty \geqslant M_L$ is obvious, hence to prove the theorem it is enough to show that $M_L^\infty \leqslant M_L$.

Assume that $M_L^\infty > M_L$. Let $z = z(t) \not\equiv 0$ be the solution of (4), (5) with $p(t)$ summable on $[0, h]$ satisfying

$$\int_0^h p(t)\,dt < M_L^\infty. \tag{12}$$

By Lemma 2, there are sequences $\{p_n\}, \{x_n\} \subset C^\infty_{[0,h]}$ such that l.i.m. $p_n = p$, $\lim_{n\to\infty} ||x_n - z|| = 0$, $Lx_n = 0$. Using the identities $z = z + x_n - x_n$, $p = p_n + p - p_n$, after simple calculation we obtain from (4) the inequality

$$|x_n'| \leqslant p_n|x_n| + |p_n - p|\,|x_n| + p|z - x_n| + |z' - x_n'|$$

or, putting

$$g_n = |p_n - p|\,|x_n| + p|z - x_n| + |z' - x_n'|,$$

the inequality

$$|x_n'| \leqslant p_n|x_n| + g_n.$$

By (12), for n sufficiently large,

$$\int_0^h p_n(t)\,dt < M_L^\infty.$$

Hence, by Lemma 1,

$$|x_n(t)| \leqslant \Big[\Big(M_L - \int_0^h p_n(t)\,dt\Big)^{-1} \exp\Big(\int_0^h p_n(t)\,dt\Big) + \exp\Big(\int_0^h p_n(t)\,dt\Big)\Big] \int_0^h g_n(t)\,dt.$$

Since the expression in the square brackets is bounded and $\lim_{n\to\infty} \int_0^h g_n(t)\,dt = 0$, we have $\lim_{n\to\infty} ||x_n|| = 0$, contrary to the assumption $||z|| > 0$.

From Theorem 4 we obtain immediately the following

Corrolary. *Theorems 2 and 3 remain true if condition (7) imposed on $p(t)$ is replaced by the inequality*

$$\int_0^h p(t)\,dt < M_L^\infty.$$

5. From Theorems 2 and 3 one can obtain optimal estimates assuring the existence and uniqueness of Nicoletti's boundary value problem [7] or Floquet's boundary value problem [3]. In this section we give further generalization of results of [3].

Theorem 5. *Let the mapping* $L: C_{[0,h]} \to R^n$ *be defined by the formula*

$$Lx = x(0) + \lambda Bx(h),$$

where $\lambda > 0$ *and the matrix* B *is orthogonal. Denote by* $a(u)$ *the angle between* u *and* Bu $(u \in R^n)$.

If $-a \leqslant a(u) \leqslant a$ *for all* u, *then*

$$M_L = \sqrt{(\pi - a)^2 + \ln^2 \lambda}.$$

Proof. Let $x = x(t)$ satisfy (4) and the boundary condition

$$x(0) + \lambda Bx(h) = 0. \tag{13}$$

From the Gronwall's inequality ([2], p. 24) it follows that if $x(t_0) = 0$ for $t_0 \in [0, h]$, then $x(t) \equiv 0$. So we can assume that $x(t) \neq 0$ on $[0, h]$.

Choose $\varepsilon > 0$ so small that $p^*(t) = p(t) + \varepsilon$ satisfies (4). A change of the independent variable $\tau = \int_0^t p^*(s)\,ds$ transforms (4) and (13) into

$$|y'(\tau)| \leqslant |y(\tau)|, \tag{14}$$

$$y(0) + \lambda By(h_1) = 0, \quad \left(h_1 = \int_0^h p^*(s)\,ds\right). \tag{15}$$

Put $r(\tau) = |y(\tau)|$, $e(\tau) = y(\tau)/|y(\tau)|$. Then (14) and (15) give

$$\left(r'(\tau)\right)^2 + \left(r(\tau)\right)^2 |e'(\tau)|^2 \leqslant \left(r(\tau)\right)^2, \tag{16}$$

$$r(0) = \lambda r(h_1). \tag{17}$$

Since the angle between $e(0)$ and $e(h_1)$ is less then $\pi - a$,

$$\int_0^{h_1} |e'(s)|\,ds \geqslant \pi - a. \tag{18}$$

Using the inequality

$$\int_0^\tau \sqrt{1 - \left(r'(s)/r(s)\right)^2}\,ds \leqslant \sqrt{\tau^2 - \ln^2\left(r(\tau)/r(0)\right)}$$

(see [3]) and the inequality

$$|e'(\tau)| \leqslant \sqrt{1 - \left(r'(\tau)/r(\tau)\right)^2}$$

we obtain

$$\pi - a \leqslant \sqrt{h_1^2 - \ln^2\left(r(h_1)/r(0)\right)}.$$

So

$$\sqrt{(\pi - a)^2 + \ln^2 \lambda} \leqslant h_1 = \int_0^h p^*(s)\,ds,$$

which contradicts (13).

An immediate consequence of Theorems 2, 3 and 5 is the following

Theorem 6. *If $f(t, x)$ satisfies (C) and (10), $p(t)$ satisfies*

$$\int_0^h p(t)\,dt < \sqrt{(\pi-a)^2+\ln^2\lambda}\,, \tag{19}$$

then problem (1), (13) has exactly one solution.

The estimation (19) is optimal, i.e., it cannot be replaced by the weak inequality

$$\int_0^h p(t)\,dt \leqslant \sqrt{(\pi-a)^2+\ln^2\lambda}\,. \tag{20}$$

To prove this, consider the system of differential equations

$$x_1' = -x_1(2/\pi)\ln\lambda + x_2\,, \qquad x_2' = -x_1 - x_2(2/\pi)\ln\lambda \tag{21}$$

with boundary conditions

$$x_1(0)+\lambda x_2(h) = 0\,, \qquad x_2(0)-\lambda x_1(h) = 0\,. \tag{22}$$

The right-hand sides of (21) satisfy (10) with

$$p(t) = \frac{2}{\pi}\sqrt{\left(\frac{\pi}{2}\right)^2+\ln^2\lambda}\,.$$

Let $h=\pi/2$. Obviously, $p(t)$ satisfies (20).

It is easy to verify that the general solution of (21),

$$x_1(t) = \exp[(-2t/\pi)\ln\lambda](C_1\cos t + C_2\sin t)\,,$$

$$x_2(t) = \exp[(-2t/\pi)\ln\lambda](-C_1\sin t + C_2\cos t)\,,$$

satisfies (22) for arbitrary C_1, C_2 while the inhomogeneous boundary value problem

$$x_1(0)+\lambda x_2(\pi/2) = r_1\,, \qquad x_2(0)-\lambda x_1(\pi/2) = r_2 \qquad (r_1^2+r_2^2>0)$$

has, in general, no solution.

REFERENCES

[1] R. Conti, *Problèmes linéaires pour les équations différentielles ordinaires*, Math. Nachr. 23 (1961), 161—178.

[2] P. Hartman, *Ordinary differential equations*, New York 1964.

[3] S. Kasprzyk, J. Myjak, *On the existence of solutions of the Floquet problem for ordinary differential equations*, Zeszyty Naukowe UJ, Prace Matematyczne 13 (1969), 35—39.

[4] А. И. Красносельский, *Топологические методы в теории нелинейных интегральных уравнений*, Москва 1956.

[5] A. Lasota, *Sur l'existence de solutions des problèmes linéaires aux limites pour les équations différentielles ordinaires*, Zeszyty Naukowe UJ, Prace Matematyczne 10 (1965), 45—60.

[6] A. Lasota, *Une généralisation du premier théorème de Fredholm et ses applications à la théorie des équations différentielles ordinaires*, Ann. Polon. Math. 18 (1966), 65—78.

[7] A. Lasota, C. Olech, *An optimal solution of Nicoletti's boundary value problem*, Ann. Polon. Math. 18 (1966), 131—140.

[8] A. Lasota, Z. Opial, *On the existence of solutions of linear problems for ordinary differential equations*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 14 (1966), 371—376.

[9] A. Pliś, *Measurable orientor fields*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys. 13 (1965), 565—569.