ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCLII PRACE MATEMATYCZNE, ZESZYT 15 1971

Anna Krakowiak, Krystyna Sewiol

On the Continuous Dependence of Solutions of a Boundary Value Problem for Ordinary Second-Order Differential Equations

1. It is well known that for an ordinary differential equation the uniqueness of solutions to initial value problems implies the continuous dependence of solutions on initial values. It is natural to ask if a similar result is true for boundary value problems. In the present paper we show that for a broad class of second-order differential equations the answer to this question is affirmative.

2. Consider a second-order differential equation

$$x'' = f(t, x, x') \tag{1}$$

and assume that the real function $f(t, x, u)$ is continuous on $I \times R^2$ where I denotes an interval of the real line R. We assume moreover that

(i) For every $(t_0, x_0, x_0') \in I \times R^2$, the initial value problem (1) and

$$x(t_0) = x_0, \quad x'(t_0) = x_0' \tag{2}$$

has a unique solution $x(t)$ defined in I, $x(t) \in C^2(I)$.

(ii) For every pair (a, α), (b, β) of points of $I \times R$, the boundary value problem (1) and

$$x(a) = \alpha, \quad x(b) = \beta \quad (a < b) \tag{3}$$

has at most one solution.

Under the above assumptions (i) and (ii) we have the following

Theorem 1. *If* $\{x_n(t)\}$ $(n = 1, 2, \ldots)$ *is the solution of equation* (1) *satisfying*

$$x_n(a_n) = \alpha_n, \quad x_n(b_n) = \beta_n \tag{3n}$$

and if

$$(a_n, \alpha_n) \to (a, \alpha), \quad (b_n, \beta_n) \to (b, \beta) \tag{4}$$
$$\big((a_n, \alpha_n), (b_n, \beta_n), (a, \alpha), (b, \beta) \in I \times R\big)$$

as $n \to +\infty$, *then the sequence* $\{x_n(t)\}$ *is almost uniformly convergent on* I *and the limit*

$$x(t) = \lim_{n \to +\infty} x_n(t)$$

is the solution of equation (1) *satisfying condition* (3) *with* (a, α) *and* (b, β) *given by* (4).

Proof. The existence of solutions of problem (1), (3) follows [1] from the assumptions (i) and (ii).
Denote by $X_1(t)$ the solution of equation (1) satisfying the condition

$$X_1(a) = \alpha + 1, \quad X_1(b) = \beta + 1$$

and by $X_2(t)$ the solution of equation (1) such that

$$X_2(a) = \alpha - 1, \quad X_2(b) = \beta - 1$$

From assumptions (ii) and (4) it follows that, for any fixed positive number $\varepsilon < \frac{1}{2}(b-a)$, there exists an integer N such that

$$X_2(t) \leqslant x_n(t) \leqslant X_1(t) \quad (a+\varepsilon \leqslant t \leqslant b-\varepsilon;\ n \geqslant N).$$

We have therefore

$$\frac{X_2(b-\varepsilon) - X_1(a+\varepsilon)}{b-a-2\varepsilon} \leqslant \frac{x_n(b-\varepsilon) - x_n(a+\varepsilon)}{b-a-2\varepsilon} \leqslant \frac{X_1(b-\varepsilon) - X_2(a+\varepsilon)}{b-a-2\varepsilon} \quad (n \geqslant N).$$

Setting

$$s_n = \min\left\{t : a+\varepsilon \leqslant t \leqslant b-\varepsilon,\ x_n'(t) = \frac{x_n(b-\varepsilon) - x_n(a+\varepsilon)}{b-a-2\varepsilon}\right\},$$

we have

$$a+\varepsilon \leqslant s_n \leqslant b-\varepsilon,$$

$$\min\{X_2(t) : t \in [a+\varepsilon, b-\varepsilon]\} \leqslant x_n(s_n) \leqslant \max\{X_1(t) : t \in [a+\varepsilon, b-\varepsilon]\},$$

$$\frac{X_2(b-\varepsilon) - X_1(a+\varepsilon)}{b-a-2\varepsilon} \leqslant x_n'(s_n) \leqslant \frac{X_1(b-\varepsilon) - X_2(a+\varepsilon)}{b-a-2\varepsilon}.$$

Replacing, if necessary, the sequence $\{s_n\}$ by a subsequence, we may assume that the limits

$$s_0 = \lim_{n \to \infty} s_n, \quad x_0 = \lim_{n \to \infty} x_n(s_n), \quad x_0' = \lim_{n \to \infty} x_n'(s_n)$$

exist. From the continuous dependence of solutions of equation (1) on their initial values it follows that the sequence $\{x_n(t)\}$ is almost uniformly convergent on I to a solution $x(t)$ of equation (1) such that $x(s_0) = x_0$ and $x'(s_0) = x_0'$. Moreover, by (3n) and (4) we have

$$x(a) = \alpha, \quad x(b) = \beta$$

and this completes the proof of Theorem 1.

3. The existence of solutions of the boundary-value problem (1), (3), deduced from [1], played an important part in the proof of Theorem 1. There is another

situation [2], more general than that covered by Theorem 1, in which the existence of solutions of problem under consideration is guaranteed by their uniqueness. We shall show that also in that case we have the continuous dependence of solutions on their boundary-values.

As in [2], consider a sequence (finite or infinite) of intervals $[e_j, u_j]$ $(j = 1, \ldots, k,$ $k \leqslant +\infty;\ 0 < e_1 \leqslant u_1 < e_2 \leqslant u_2 < \ldots)$ assume that $u_j - e_j < e_1$ $(j = 1, 2, \ldots)$ and suppose that

(iii) If two distinct integral curves of equation (1) agree in value at $t_0 \in I$, then if they agree in value again at z, there is a positive integer i such that

$$e_i \leqslant |t_0 - z| \leqslant u_i ,$$

and if u_k is finite, then for each $j = 1, \ldots, k$ there is an t_j such that the curves agree in value at t_j and

$$e_j \leqslant |t_0 - t_j| \leqslant u_j .$$

Under assumptions (i) and (iii) we have the following

Theorem 2. *If* $\{x_n(t)\}$ $(n = 1, 2, \ldots)$ *is a solution of equation* (1) *satisfying*

$$x_n(a_n) = a_n , \quad x_n(b_n) = \beta_n , \tag{5}$$

if moreover

$$(a_n, \alpha_n) \to (a, \alpha) , \quad (b_n, \beta_n) \to (b, \beta) \quad \big((a_n, \alpha_n), (b_n, \beta_n), (a, \alpha), (b, \beta) \in I \times R\big) \tag{6}$$

as $n \to \infty$ *and if for some* k

$$u_{k-1} < b - a < e_k , \tag{7}$$

then the sequence $\{x_n(t)\}$ *is almost uniformly convergent on* I *and its limit,*

$$x(t) = \lim_{n \to \infty} x_n(t) ,$$

is the solution of equation (1) *satisfying the condition*

$$x(a) = \alpha , \quad x(b) = \beta . \tag{8}$$

Proof. From assumptions (i), (iii) and (7), by [2], it follows the existence of solution of problem (1), (8).

When $k = 1$ Theorem 2 reduces to Theorem 1. We shall prove Theorem 2 when $k = 2$ only. It will be easily seen that the general case follows in the same way.

First we consider the case $a_n = a$, $\alpha_n = \alpha$ $(n = 1, 2, \ldots)$ and $(b_n, \beta_n) \to (b, \beta)$. Denote by $X_1(t)$ the solution of equation (1) satisfying the condition

$$X_1(a) = \alpha , \quad X_1(b) = \beta + 1$$

and by $X_2(t)$ the solution of equation (1) satisfying the condition

$$X_2(a) = \alpha , \quad X_2(b) = \beta - 1 .$$

From (6) and (iii) it follows that

$$X_1(t) \leqslant x_n(t) \leqslant X_2(t) \quad (n \geqslant N,\ a \leqslant t \leqslant e_1).$$

As in the proof of Theorem 1 we find a sequence of points $\{s_n\}$ $(a \leqslant s_n \leqslant e,$ $n = 1, 2, \ldots)$ such that the sequences $\{x_n(s_n)\}$ and $\{x_n'(s_n)\}$ are bounded. Hence, by (i), it follows that the sequence $\{x_n(t)\}$ is almost uniformly convergent on I to solution $x(t)$ of equation (1) satisfying the condition (8).

The proof in the case when $b_n = b$, $\beta_n = \beta$ $(n = 1, 2, \ldots)$ and $(a_n, \alpha_n) \to (a, \alpha)$ is quite analogous.

Finally, applying the triangle inequality we obtain the complete proof of Theorem 2.

4. By a standard compactness argument one can show that the continuous dependence just proved is almost uniformly in the following sense: for any compact subset K of R^2, any compact subinterval $\varDelta$ of I and for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $P_1(a_1, \alpha_1)$, $P_2(a_2, \alpha_2)$, $Q_1(b_1, \beta_1)$, $Q_2(b_2, \beta_2)$ $(P_1 \neq Q_1,\ P_2 \neq Q_2)$ are arbitrary points of $\varDelta \times K$ and $\varrho(P_1, P_2) < \delta$, $\varrho(Q_1, Q_2) < \delta$ then $|x_1(t) - x_2(t)| + |x_1'(t) - x_2'(t)| < \varepsilon$ $(t \in \varDelta)$ where $x_1(t), x_2(t)$ denote the solutions of equation (1) satisfying, respectively, the conditions $x_1(a_1) = \alpha_1$, $x_1(b_1) = \beta_1$, $x_2(a_2) = \alpha_2$, $x_2(b_2) = \beta_2$.

REFERENCES

[1] A. **Lasota**, Z. **Opial**, *On the existence and uniqueness of solutions of a boundary value problem for an ordinary second order differential equation*, Colloq. Math. 18 (1967), 7—11.

2] L. F. **Shampine**, *Existence and uniqueness for nonlinear boundary value problems*, J. of Diff. Equations 5 (1969), 346—351.