

Andrzej Lasota, Zdzisław Opial

On a Modified Isochronism Problem for the Non-linear Oscillator

1. The well-known property of the linear second-order differential equation

$$x'' + \omega^2 x = 0 \quad (\omega > 0)$$

that all its solutions are periodic with the same period $2\pi/\omega$ has the counterpart in the theorem ([1], p. 351) stating that, conversely, if all solutions of the equation

$$x'' + h(x) = 0 \quad (h(x) = -h(-x), xh(x) > 0 \text{ for } x \neq 0)$$

are periodic with the same period $2\pi/\omega$ then $h(x) = \omega^2 x$.

More generally, for two non-linear second-order differential equations

$$(1) \quad x'' + h_1(x) = 0 \quad (h_1(x) > 0 \text{ for } x \in (0, a], h_1(0) = 0)$$

and

$$(2) \quad x'' + h_2(x) = 0 \quad (h_2(x) > 0 \text{ for } x \in (0, a], h_2(0) = 0)$$

a similar property may be stated as follows: if, for every $\xi \in [0, a]$, the solutions $x_1(t, \xi)$, $x_2(t, \xi)$ of equations (1) and (2), respectively, satisfying the initial-value conditions

$$(3) \quad x(0) = \xi, \quad x'(0) = 0$$

reach the equilibrium point $x = 0$ in the same time $T(\xi)$, then $h_1(x) = h_2(x)$ in $[0, a]$. As shown in [4], the requirement of continuity is the only smoothness condition on $h_1(x)$ and $h_2(x)$ needed in the proof of this theorem (for further generalizations see [3] and [5]).

2. One may modify the above procedure by taking the solutions of equations (1) and (2) satisfying, in the place of (3), the following initial-value conditions

$$x(0) = a, \quad x'(0) = \eta \quad (-\infty < \eta < 0)$$

(this means that one fixes the starting point and varies the velocity) and then raise a similar problem with a clear mechanical significance: does the equality of the times of reaching the equilibrium point $x = 0$ imply the identity of the functions $h_1(x)$ and $h_2(x)$? A positive answer to this problem is given by the following

Theorem. *If the continuous functions $h_i(x)$ ($i = 1, 2$) are positive for $x \in (0, a]$ and if for a sequence $\{\eta_k\}$ of negative real numbers such that*

$$(4) \quad \lim_{k \rightarrow \infty} \eta_k = \eta_0 < 0$$

the solutions of equations (1) and (2) satisfying the initial-value conditions

$$x(0) = a, \quad x'(0) = \eta_k \quad (k = 1, 2, \dots)$$

reach the equilibrium point $x = 0$ in the same time T_k , then $h_1(x) = h_2(x)$ in $[0, a]$.

Proof. Upon setting

$$r_i(x) = 2 \int_x^a h_i(s) ds \quad (i = 1, 2)$$

we have (see, for instance, [3])

$$T_k = \int_0^a \frac{dx}{\sqrt{\eta_k^2 + r_i(x)}} \quad (i = 1, 2; k = 1, 2, \dots)$$

Hence and from (4) it follows that the analytic functions

$$\varphi_i(z) = \int_0^a \frac{dx}{\sqrt{z^2 + \eta_0^2 + r_i(x)}} \quad (i = 1, 2)$$

are defined and identical in the circle $|z| < |\eta_0|$. Expanding these functions into power series and putting, for to simplify,

$$v_i(x) = (\eta_0^2 + r_i(x))^{-1/2} \quad (i = 1, 2)$$

we get

$$\begin{aligned} \varphi_1(z) &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} z^{2n} \int_0^a [v_1(x)]^{2n+1} dx \\ &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} z^{2n} \int_0^a [v_2(x)]^{2n+1} dx = \varphi_2(z) \quad (|z| < |\eta_0|). \end{aligned}$$

Hence

$$\int_0^a [v_1(x)]^{2n+1} dx = \int_0^a [v_2(x)]^{2n+1} dx \quad (n = 1, 2, \dots).$$

It is easily seen that for to complete the proof it suffices to show that $v_1(x) = v_2(x)$ in $[0, a]$. Thus, the functions $v_1(x)$, $v_2(x)$ being continuous, nonnegative and non-increasing, the proof of our Theorem reduces to a straightforward application of the following

Lemma. Given an arbitrary sequence $\{\alpha_n\}$ of nonnegative real numbers, there exists in the interval $[0, a]$ at most one continuous, nonnegative and nonincreasing function $v(x)$ such that

$$\int_0^a [v(x)]^{2n+1} dx = \alpha_n \quad (n = 1, 2, \dots).$$

Proof. If such a function exists then, as it is well-known, the sequence $\{\sqrt[2n+1]{\alpha_n}\}$ converges to $\max\{v(x) : x \in [0, a]\} = v(0)$. Thus the value

$$v(0) = \lim_{n \rightarrow \infty} \sqrt[2n+1]{\alpha_n} = a$$

is well determined by the given sequence $\{\alpha_n\}$.

It is easy to show (by a simple geometrical proof which may be left to the reader) that the function $v(x)$ is uniquely determined by the function

$$q(y) = \int_0^a \min\{y, v(x)\} dx,$$

continuous and increasing in the interval $[0, a]$. Therefore, for to complete the proof, it is sufficient to show that the function $q(y)$ is uniquely determined by the sequence $\{\alpha_n\}$.

To this end observe first that setting, for $y, u \geq 0$,

$$p(y, u) = \begin{cases} u & \text{for } 0 \leq u \leq y \\ y & \text{for } y \leq u \end{cases}$$

we have $\min\{y, v(x)\} = p(y, v(x))$ and therefore

$$q(y) = \int_0^a p(y, v(x)) dx \quad (0 \leq y \leq a).$$

Choose now, for a fixed $y \in [0, a]$, a sequence of polynomials

$$w_n(u) = y + \sum_{k=0}^n a_{nk} u^{2k+1} \quad (n = 1, 2, \dots)$$

converging uniformly in $[0, a]$ to the continuous function $p(y, u)$ (such a sequence exists by the well-known Müntz refinement of the classical Weierstrass theorem, see for instance [2]). We have

$$q(y) = \int_0^a p(y, v(x)) dx = \lim_{n \rightarrow \infty} \int_0^a w_n(v(x)) dx = \lim_{n \rightarrow \infty} \left(ay + \sum_{k=0}^n a_{nk} \alpha_k \right)$$

which means that the value $q(y)$ is well determined by the sequence $\{\alpha_n\}$ and completes the proof.

3. One may also give an alternative, more sophisticated proof of the Lemma, based on the theory of the Hausdorff problem of moments. To this end it suffices to observe that, by the definition of the function $q(y)$, we have

$$(5) \quad \alpha_n = \int_0^a [v(x)]^{2n+1} dx = \int_0^a y^{2n+1} dq(y) \quad (n = 1, 2, \dots).$$

Since the Hausdorff problem of moments is determined (see for instance [2] or [6], Corollary 1.1), there exists at most one continuous increasing function $q(y)$ satisfying (5).

4. It is easy to give a simple example showing that our Theorem is no longer true if the functions $h_1(x)$, $h_2(x)$ are not assumed to be positive. For instance, setting $a = 2\pi$ and

$$h_1(x) = \frac{1}{2} \cos x, \quad h_2(x) = -\frac{1}{2} \cos x$$

for the times of reaching the point $x = 0$ by the solutions of equations (1) and (2), respectively, we have (for $\eta < -1$) the explicit formulae

$$T_1(\eta) = \int_0^{2\pi} \frac{dx}{\sqrt{\eta^2 + \sin x}}, \quad T_2(\eta) = \int_0^{2\pi} \frac{dx}{\sqrt{\eta^2 - \sin x}}$$

from which it follows immediately that $T_1(\eta) = T_2(\eta)$.

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