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On a Modified Isochronism Problem for the Non-linear Oscillator

1. The well-known property of the linear second-order differential equation

$$x''+\omega^2 x=0 \quad (\omega>0)$$

that all its solutions are periodic with the same period $2\pi/\omega$ has the counterpart in the theorem ([1], p. 351) stating that, conversely, if all solutions of the equation

$$x''+h(x)=0 \quad (h(x)=-h(-x),\ xh(x)>0 \text{ for } x\neq 0)$$

are periodic with the same period $2\pi/\omega$ then $h(x)=\omega^2 x$.

More generally, for two non-linear second-order differential equations

(1) $$x''+h_1(x)=0 \quad (h_1(x)>0 \text{ for } x\in(0,a],\ h_1(0)=0)$$

and

(2) $$x''+h_2(x)=0 \quad (h_2(x)>0 \text{ for } x\in(0,a],\ h_2(0)=0)$$

a similar property may be stated as follows: if, for every $\xi\in[0,a]$, the solutions $x_1(t,\xi)$, $x_2(t,\xi)$ of equations (1) and (2), respectively, satisfying the initial-value conditions

(3) $$x(0)=\xi, \quad x'(0)=0$$

reach the equilibrium point $x=0$ in the same time $T(\xi)$, then $h_1(x)=h_2(x)$ in $[0,a]$. As shown in [4], the requirement of continuity is the only smoothness condition on $h_1(x)$ and $h_2(x)$ needed in the proof of this theorem (for further generalizations see [3] and [5]).

2. One may modify the above procedure by taking the solutions of equations (1) and (2) satisfying, in the place of (3), the following initial-value conditions

$$x(0)=a, \quad x'(0)=\eta \quad (-\infty<\eta<0)$$

(this means that one fixes the starting point and varies the velocity) and then raise a similar problem with a clear mechanical significance: does the equality of the times of reaching the equilibrium point $x=0$ imply the identity of the functions $h_1(x)$ and $h_2(x)$? A positive answer to this problem is given by the following

Theorem. *If the continuous functions* $h_i(x)$ $(i=1,2)$ *are positive for* $x \in (0, a]$ *and if for a sequence* $\{\eta_k\}$ *of negative real numbers such that*

$$\lim_{k\to\infty} \eta_k = \eta_0 < 0 \tag{4}$$

the solutions of equations (1) *and* (2) *satisfying the initial-value conditions*

$$x(0) = a\,, \quad x'(0) = \eta_k \quad (k = 1, 2, \ldots)$$

reach the equilibrium point $x = 0$ *in the same time* T_k, *then* $h_1(x) = h_2(x)$ *in* $[0, a]$.

Proof. Upon setting

$$r_i(x) = 2\int_x^a h_i(s)\,ds \quad (i = 1, 2)$$

we have (see, for instance, [3])

$$T_k = \int_0^a \frac{dx}{\sqrt{\eta_k^2 + r_i(x)}} \quad (i = 1, 2;\ k = 1, 2, \ldots)$$

Hence and from (4) it follows that the analytic functions

$$\varphi_i(z) = \int_0^a \frac{dx}{\sqrt{z^2 + \eta_0^2 + r_i(x)}} \quad (i = 1, 2)$$

are defined and identical in the circle $|z| < |\eta_0|$. Expanding these functions into power series and putting, for to simplify,

$$v_i(x) = \left(\eta_0^2 + r_i(x)\right)^{-1/2} \quad (i = 1, 2)$$

we get

$$\varphi_1(z) = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} z^{2n} \int_0^a [v_1(x)]^{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} z^{2n} \int_0^a [v_2(x)]^{2n+1} dx = \varphi_2(z) \quad (|z| < |\eta_0|)\,.$$

Hence

$$\int_0^a [v_1(x)]^{2n+1} dx = \int_0^a [v_2(x)]^{2n+1} dx \quad (n = 1, 2, \ldots)\,.$$

It is easily seen that for to complete the proof it suffices to show that $v_1(x) = v_2(x)$ in $[0, a]$. Thus, the functions $v_1(x)$, $v_2(x)$ being continuous, nonnegative and non-increasing, the proof of our Theorem reduces to a straightforward application of the following