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# A Metric Connection on the Null Hypersurface and Discussion of Treder's Parallel Transport

## I. INTRODUCTION

Let $L(M_n)$ be the principal fibre bundle of linear frames over a connected, $n$-dimensional metric manifold [1]) $M_n$ of the class $C^\infty$. Components $\Gamma^\alpha_{\beta\gamma}$ of the linear metric connection in the bundle $L(M_n)$ with respect to a local coordinate system on the $M_n$ satisfy locally the equations [14], [16]

$$(1)\qquad \nabla_\alpha g_{\beta\gamma} \equiv \partial_\alpha g_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} g_{\delta\gamma} - \Gamma^\delta_{\alpha\gamma} g_{\beta\delta} = 0$$
$$\alpha, \beta, \gamma, \delta, \varepsilon = 1, \ldots, n,$$

where $g_{\alpha\beta}$ is the metric tensor of the $M_n$.

Equations (1) permit the components $\Gamma^\alpha_{\beta\gamma}$ to be calculated explicitly for a given symmetric tensor $g_{\alpha\beta}$ and, therefore, a metric connection $\Gamma$ in the bundle $L(M_n)$ [14]. We can transform them [24] to the more convenient form

$$(2)\qquad \Gamma^\alpha_{\beta\gamma} g_{\delta\alpha} = [\beta\gamma, \delta] - g_{\gamma\alpha} S^\alpha_{\delta\beta} - g_{\beta\alpha} S^\alpha_{\delta\gamma} + g_{\alpha\delta} S^\alpha_{\beta\gamma},$$

where $S^\alpha_{\beta\gamma}$ is the torsion tensor.

If the metric of $M_n$ is non-singular, i.e. if $M_n$ is $V_n$ then there exists a tensor $g^{\alpha\beta}$ such that

$$(3)\qquad g^{\alpha\beta} g_{\beta\delta} = \delta^\alpha_\delta .$$

The contraction of $g^{\delta\varepsilon}$ with the equations (2) leads immediately to the solution

$$(4)\qquad \Gamma^\varepsilon_{\beta\gamma} = \left\{ {\varepsilon \atop \beta\gamma} \right\} + S^\varepsilon_{\beta\gamma} - g^{\delta\varepsilon}(g_{\gamma\alpha} S^\alpha_{\delta\beta} + g_{\beta\alpha} S^\alpha_{\delta\gamma})$$

[1]) We restrict ourselves to connected manifolds and fields both of class $k = \infty$ (exception: a null hypersurface immersed in space-time and fields on it). However, the results for $n = 3$ are true for the values of $k$ demanded in general relativity. This is immediately evident from Section II and from Section V.

We see from (4) that symmetric, non-degenerate metric and torsion determine a unique metric connection in the bundle $L(V_n)$. We conclude also that a symmetric and non-singular metric $g_{\alpha\beta}$ determines uniquely a metric and symmetric linear connection $\Gamma$ in this bundle. The components of this connection with respect to a local coordinate system in the $V_n$ are Christoffel symbols of the second kind. In this paper we shall express a linear connection and related concepts mostly in terms of the local coordinate system (exception: Section III).

$n$-dimensional metric manifold with a metric tensor $g_{\alpha\beta}$ such that $\det\|g_{\alpha\beta}\| = 0$ is called a degenerate or singular Riemannian manifold. The number $r$ such that $(n-r)$ is the constant rank of the matrix $\|g_{\alpha\beta}\|$ determines the degree of degeneracy. In $r$-time degenerate, $n$-dimensional Riemannian manifold, which will be denoted by $V_n^{(r)}$, there exist $r$ linearly independent vector fields $\underset{A}{k}{}^{\alpha}$, $A = (n-r+1), \ldots, n$ such that at every point of the manifold

$$g_{\alpha\beta}\underset{A}{k}{}^{\alpha} = 0 . \tag{5}$$

We shall call the vectors $\underset{A}{k}{}^{\alpha}(p)$, at the point $p$, the null vectors of the metric $g_{\alpha\beta}$ at the point $p$. In the manifold $V_n^{(r)}$ the tensor $g^{\alpha\beta}$, which satisfies the equations (3), does not exist.

If there exists in $V_n^{(r)}$ $(1 \leqslant r \leqslant n-1)$ a coordinate system in which the tensor $g_{\alpha\beta}$ has the canonical form

$$\|g_{\alpha\beta}\| = \begin{Vmatrix} g_{\lambda\mu} & 0 \\ 0 & 0 \end{Vmatrix}, \quad \det\|g_{\lambda\mu}\| \neq 0; \tag{6}$$

$$\lambda, \mu, \nu, \pi, \varrho, \sigma, \tau = 1, \ldots, (n-r),$$

then $V_n^{(r)}$ is called reducible. If there is a coordinate system in which the metric has the canonical form and $\partial_A g_{\lambda\mu} = 0$, then $V_n^{(r)}$ is called $r$-time and absolutely reducible [28]. The coordinate system in which the metric has the form (6) is called the canonical coordinate system.

A particular example of $V_3^{(1)}$ is the null hypersurface $\overset{*}{V}_3$ in space-time of the general theory of relativity. The interior metric of such a surface is degenerate and we cannot solve the equations (1) in the usual way. The metric degeneracy of the null hypersurface is inseparably joined with a lack of the normal rigging of such a hypersurface. Normal vectors to the null hypersurface $\overset{*}{V}_3$ are simultaneously tangent to it. These vectors determine the congruence of null geodesics in space-time filling completely $\overset{*}{V}_3$. The geodesics of this congruence are bicharacteristics of Einstein's field equations, while the null hypersurface $\overset{*}{V}_3$ is the characteristic manifold of these equations [18], [19], [23].

Many authors have examined the internal geometry of this hypersurface, e.g. [3], [4], [5], especially the problem of the internal linear connection determined by its immersion in space-time and the equivalent problem [1] of the internal parallel transport on it. Lemmer [15] used the method of projection of the affinity from space-time by means of rigging of the hypersurface.

The affine connection obtained in this way is not metric and is not uniquely determined because there is no natural rigging [8]. The procedure of Daŭtcourt [5] is similar to ours, however the class of connections which he obtains is metric only on an absolutely reducible hypersurface $\overset{*}{V}_3$. The same class of connections on the null hypersurface $\overset{*}{V}_3$ was examined by Kemmerer [12]. Treder [26] proposes parallel transport on the null hypersurface which uses only Christoffel's symbols of the first kind which are uniquely determined.

We shall show in Section V that Treder's proposition is a usual parallel transport only on the absolutely reducible $\overset{*}{V}_3$ and that, in this case, it corresponds to infinitely many metric and symmetric linear connections.

The problem of the metric connection in the general case of $V_n^{(r)}$ has been examined by E. Bortolotti [2] (absolutely reducible $V_n^{(r)}$) and by O. Vogel [28]. See also the papers of A. Jakubowicz [10] and C. Jankiewicz [11].

## II. METRIC CONNECTION ON A NULL HYPERSURFACE $\overset{*}{V}_n$

We shall consider the case of an $n$-dimensional null hypersurface $\overset{*}{V}_n$, immersed in a Riemannian manifold $V_{n+1}$ with a sygnature $(++, ..., +-)$. Such a hypersurface is a particular example of $V_n^{(1)}$. O. Vogel has proved two important theorems concerning linear metric connection in $V_n^{(r)}$.

Theorem 1 [28]. $V_n^{(r)}(1 \leqslant r \leqslant n-1)$ admits of metric and symmetric linear connection if and only if it is $r$-time and absolutely reducible.

Theorem 2 [28]. Equations (1) always admit of solutions $\Gamma^{\alpha}_{\beta\gamma} \neq \Gamma^{\alpha}_{\gamma\beta}$, i.e. the non-symmetric metric connection on the $V_n^{(r)}(1 \leqslant r \leqslant n-1)$.

The analysis of equations (1) and their solutions can, therefore, be divided in a natural manner into two cases:

a) an absolutely reducible $\overset{*}{V}_n$,

b) a general $\overset{*}{V}_n$.

### a) Absolutely Reducible $\overset{*}{V}_n$.

We shall prove Theorem 1. The proof is in fact the solution of the problem for this case. We shall show first that $\overset{*}{V}_n$, which admits of a symmetric and metric linear connection, is absolutely reducible.

Every $\overset{*}{V}_n$ is reducible, i.e. admits of the canonical coordinate system in which $g_{\alpha n} = g_{n\alpha} = 0$ and $k^{\alpha} = \delta^{\alpha}_n$, where $k^{\alpha}$ are the components of the null vector, satisfying equations

$$g_{\alpha\beta} k^{\alpha} = 0 . \tag{7}$$

Let us consider in this coordinate system the equations

$$(8) \qquad \nabla_a k^\beta = \partial_a k^\beta + \Gamma^\beta_{a\gamma} k^\gamma = l_a k^\beta ,$$

which follow from (7) and from the metric character of the connection [17].

In the canonical system

$$(9) \qquad \nabla_a k^\beta = l_a \delta^\beta_n$$

and therefore

$$(9') \qquad \Gamma^\lambda_{an} = \Gamma^\lambda_{na} = 0 .$$

Using (9′) we have

$$(10) \qquad \nabla_n g_{\lambda\mu} = \partial_n g_{\lambda\mu} - \Gamma^\nu_{n\lambda} g_{\nu\mu} - \Gamma^\nu_{n\mu} g_{\lambda\nu} = \partial_n g_{\lambda\mu} = 0$$

which means that the $\overset{*}{V}_n$ is absolutely reducible.

We shall now prove Theorem 1 in the opposite direction. Solutions of the equations (1) can be obtained most easily in the canonical coordinate system. We have

$$(11.1) \qquad \nabla_\lambda g_{\mu\nu} \equiv \partial_\lambda g_{\mu\nu} - \Gamma^\varrho_{\lambda\mu} g_{\varrho\nu} - \Gamma^\varrho_{\lambda\nu} g_{\mu\varrho} = 0 ,$$

$$(11.2) \qquad \nabla_n g_{\mu\nu} \equiv \partial_n g_{\mu\nu} - \Gamma^\varrho_{n\mu} g_{\varrho\nu} - \Gamma^\varrho_{n\nu} g_{\mu\varrho} = 0 ,$$

$$(11.3) \qquad \nabla_a g_{\mu n} \equiv \partial_a g_{\mu n} - \Gamma^\gamma_{a\mu} g_{\gamma n} - \Gamma^\gamma_{an} g_{\mu\gamma} = 0 ,$$

$$(11.4) \qquad \nabla_a g_{nn} \equiv \partial_a g_{nn} - \Gamma^\beta_{an} g_{\beta n} - \Gamma^\beta_{an} g_{n\beta} = 0 .$$

Eq. (11.4) is fulfilled identically, because in the canonical coordinate system $g_{na} = 0$.

From (11.3), $g_{na} = 0$ and $\det\|g_{\mu\nu}\| \neq 0$ it follows that $\Gamma^\varrho_{an} = \Gamma^\varrho_{na} = 0$. Eq. (11.2) is fulfilled identically because $\Gamma^\varrho_{na} = 0$ and $\partial_n g_{\mu\nu} = 0$. Since $\det\|g_{\mu\nu}\| \neq 0$, one can solve the equations (11.1) in the usual way [24] and get $\Gamma^\lambda_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$.

It is seen that in the canonical coordinate system, the metric determines uniquely only the components $\Gamma^\lambda_{\beta\gamma} = \Gamma^\lambda_{\gamma\beta}$; the remaining $\dfrac{n(n+1)}{2}$ components $\Gamma^n_{\alpha\beta} = \Gamma^n_{\beta\alpha}$ may be chosen arbitrarily.

On applying this theorem to the case of $\overset{*}{V}_3$ we see that the metric of the absolutely reducible $\overset{*}{V}_3$ admits of a whole class of symmetric and metric linear connections. This class depends on six arbitrary functions, namely $\Gamma^3_{\alpha\beta}$ in the canonical coordinate system.

The null curves (the generators of $\overset{*}{V}_3$), tangent at every point to the null vector, are geodesics for every connection of this class.

We shall denote the class of metric and symmetric linear connections on the absolutely reducible null hypersurface $\overset{*}{V}_n$ by $[\Gamma^\alpha_{\beta\gamma}]_s$.

Having the components of the linear connection in the canonical coordinate system we define them in every other coordinate system by means of the transformation rule

$$\Gamma^{\alpha'}_{\beta'\gamma'} = \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial x^{\beta'}} \frac{\partial x^{\gamma}}{\partial x^{\gamma'}} \Gamma^{\alpha}_{\beta\gamma} + \frac{\partial x^{\alpha'}}{\partial x^{\alpha}} \frac{\partial^2 x^{\alpha}}{\partial x^{\beta'} \partial x^{\gamma'}} . \tag{12}$$

## b) The General $\overset{*}{V}_n$.

The proof of Theorem 2 is trivial if the theorem of Wong [30] is used. This theorem, in application to the metric tensor $g_{\alpha\beta}$, says that the metric connection, i.e. the solution of the system (1) exists if and only if in every point of $V_n^{(r)}$ there exists the basis [2]) $\underset{(\beta)}{e^{\alpha}}$ in which the components of the tensor $g_{\alpha\beta}$ are constant and not all equal to zero. The tensor $g_{\alpha\beta}$ admitting of such a field $\underset{(\beta)}{e^{\alpha}}$ is called homogeneous and the field $\underset{(\beta)}{e^{\alpha}}$ the regular field [30]. It is easy to see that on the $\overset{*}{V}_n$ there are infinitely many regular fields (see Section III). Therefore, the system (1) is always compatible and, by virtue of Theorem 1, the solutions are generally non-symmetric.

As in the case of absolute reducibility, one can solve the equations (1) most easily in the canonical coordinate system in which they assume the form (11). Eq. (11.4) are fulfilled identically. It results from (11.3) that $\Gamma^{\lambda}_{\alpha n} = 0$. Eq. (11.1) can be solved in the usual way [24]: $\Gamma^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$. It is more difficult to solve the system (11.2). It is the system of $\frac{n(n-1)}{2}$ equations (linear in every point) on $(n-1)^2$ unknown components $\Gamma^{\lambda}_{n\mu}$.

In the case of a null hypersurface $\overset{*}{V}_3$ in space-time, which is particularly interesting for us, this system gives three equations for four unknowns $\Gamma^1_{31}$, $\Gamma^1_{32}$, $\Gamma^2_{31}$, $\Gamma^2_{32}$. Summarizing, if the components $\Gamma^3_{\alpha\beta}$ and $\Gamma^2_{32}$ are given in the canonical coordinate system, then, in the general case of $\overset{*}{V}_3$, the metric tensor determines the remaining components of linear connection. We put here, for simplicity, $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$ (in the canonical coordinate system). In general, the number of arbitrary components increases to twelve.

The number $f$ of arbitrary components for $\overset{*}{V}_n$ equals

$$f = \frac{n^2(n-1)}{2} + n . \tag{13}$$

The special class of the non-symmetric, linear and metric connections on the $\overset{*}{V}_n$ for which, in the canonical coordinate system, $\Gamma^{\lambda}_{\mu\nu} = \{^{\lambda}_{\mu\nu}\}$ we shall denote by $[\Gamma^{\alpha}_{\beta\gamma}]_{ns}$.

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[2] The indices in round brackets always refer to a local frame.