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On Some Eigenvalues and Eigenfunctions of the Composition of two Sturm-Liouville Type Operators

1. Let $C^k([a, b])$, $k \geqslant 2$, denote a linear space of all functions of class C^k in interval $[a, b]$.

Let L_1, L_2 be operators,

$$L_j\colon\ C^k([a, b]) \to C^{k-2}([a, b])\,, \quad j = 1, 2$$

defined by

$$L_j u((t) = -(p_j(t)u')' + q_j(t)u(t)\,, \quad j = 1, 2 \tag{1}$$

where $p_j(t) > 0$, $q_j(t) \geqslant 0$ for $t \in [a, b]$, $p_j \in C^{k-1}([a, b])$, $q_j \in C^{k-2}([a, b])$, $j = 1, 2$. It follows from (1) that there exists a composition $L = L_1L_2$ if only $k \geqslant 4$.

Consider the equation

$$Lu - \lambda u = 0 \tag{2}$$

with boundary conditions

$$\alpha_1 u(a) - \alpha_2 u'(a) = 0\,, \quad \beta_1 u(b) + \beta_2 u'(b) = 0 \tag{3}$$

and

$$\begin{aligned} &\alpha_3 L_2 u(t) - \alpha_4[L_2 u(t)]' = 0 \quad \text{for } t = a\\ &\beta_3 L_2 u(t) + \beta_4[L_2 u(t)]' = 0 \quad \text{for } t = b\,, \end{aligned} \tag{4}$$

where $\alpha_1, \alpha_4, \beta_1, \beta_4$ are positive numbers and $\alpha_2, \alpha_3, \beta_2, \beta_3$ non-negative ones.

Lemma 1. If a function $y(t) \in C^4([a, b])$ is a solution of problem (2) (4), then

$$\int\limits_a^b \{p_1[(L_2y)']^2 + q_1(L_2y)^2 - \lambda(L_2y)y\}\,dt \leqslant 0\,.$$

Proof. Multiplying by L_2y both sides of equation (2) we obtain

$$-L_2y[p_1(L_2y)']' + q_1(L_2y)^2 - \lambda y L_2 y = 0\,. \tag{5}$$

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where $a_1, a_4, \beta_1, \beta_4$ are positive numbers and $a_2, a_3, \beta_2, \beta_3$ non-negative ones.

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$$\int_a^b \{p_1[(L_2 y)']^2 + q_1 (L_2 y)^2 - \lambda (L_2 y) y\}\, dt \leqslant 0\,.$$

Proof. Multiplying by $L_2 y$ both sides of equation (2) we obtain

$$-L_2 y\,[p_1 (L_2 y)']' + q_1 (L_2 y)^2 - \lambda y L_2 y = 0\,. \tag{5}$$

On the other hand, we have

$$[p_1(L_2y)'L_2y]' = [p_1'(L_2y)']'L_2y + p_1[(L_2y)']^2,$$
$$L_2y\,[p_1(L_2y)']' = \{p_1(L_2y)'L_2y - [p_1(L_2y)']^2\}'.$$

Hence, by (5), we obtain

$$[p_1(L_2y)'L_2y]' = [p_1(L_2y)']^2 + q_1(L_2y)^2 - \lambda y L_2 y\,.$$

Integrating this identity over $[a, b]$ we obtain, by (4), the statement of the lemma

Lemma 2. If $y \in C^2([a, b])$ satisfies condition (3) and $L_j y \equiv 0$ in $[a, b]$ for $j = 1$ or $j = 2$, then $y \equiv 0$ in $[a, b]$.

Proof. By assumption we have

$$(p_j y')' - q_j y = 0\,.$$

Multiplying by y and integrating over $[a, b]$ we get

$$p(b)y'(b)y(b) - p(a)y'(a)y(a) = \int_a^b [p(y')^2 + qy^2]\,dt \tag{6}$$

On the other hand, it follows from (3) that

$$p(b)y'(b)y(b) - p(a)y'(a)y(a) = -p(b)\frac{\beta_2}{\beta_1}[y'(b)]^2 - p(a)\frac{\alpha_2}{\alpha_1}[y'(a)]^2. \tag{7}$$

Suppose $y \not\equiv 0$. Then, since $p(t) > 0$ for $t \in [a, b]$, the left-hand side of (6) is positive which leads to a contradiction with (7).

2. We say that a real number μ is an eigenvalue of problem (2–4) if there exists a function $u \in C^4[a, b]$, $u \not\equiv 0$, satisfying (3), (4) and equation (2) for $\lambda = \mu$. The function u is called the eigenfunction of problem (2–4) corresponding to the eigenvalue μ.

Theorem 1. If λ is an eigenvalue of problem (2–4), then $\lambda > 0$.

Proof. Let y be an eigenfunction corresponding to λ. We consider two cases. If $\lambda = 0$, then by definition of

$$L_1 L_2 y = 0\,.$$

Hence, and by Lemma 2, $L_2 y \equiv 0$. Applying this Lemma once more we obtain $y \equiv 0$. But this contradicts with the definition of y.

Suppose now $\lambda < 0$. Then, by Lemma 2, $L_2 y \not\equiv 0$ in $[a, b]$, whence

$$\int_a^b p_1[(L_2y)']^2 dt + \int_a^b q_1 (L_2 y)^2 dt > 0\,.$$

Therefore, in order that (5) may hold true we must have

$$\int_a^b (L_2 y)\, y\, dt < 0\,. \tag{8}$$

On the other hand,

$$yL_2y = -(ypy')' + p(y')^2 + qy^2.$$

Hence, by (3), we get

$$\int_a^b yL_2y\,dt = -ypy'|_a^b + \int_a^b [(py')^2 + qy^2]\,dt \geqslant 0,$$

in contradiction with (8).

Lemma 3. If λ_1, λ_2 are eigenvalues of problem (2–4) and if y, z are eigenfunctions corresponding to λ_1, λ_2, then

$$\lambda_1 \int_a^b yL_2z\,dt = \lambda_2 \int_a^b zL_2y\,dt. \tag{8'}$$

Proof. By our assumption we have

$$(p_1L_2'y)' - q_1L_2y = -\lambda_1 y \tag{9}$$

and

$$(p_1L_2'z)' - q_1L_2z = -\lambda_2 z. \tag{10}$$

Multiplying (9) and (10) by L_2z and L_2y, respectively, and subtracting (10) from (9) we obtain

$$L_2z(p_1L_2'y)' - L_2y(p_1L_2'z)' = -\lambda_1 yL_2z + \lambda_2 zL_2y.$$

Observe that

$$[p_1(L_2zL_2'y - L_2yL_2'z)]' = L_2z(p_1L_2'y)' - L_2y(p_1L_2'z)'.$$

Therefore

$$[p_1(L_2zL_2'y - L_2yL_2'z)]' = \lambda_2 zL_2y - \lambda_1 yL_2z \tag{11}$$

Integrating (11), (4), we come to relation (8′).

3. Denote by A a subclass of $C^2([a, b])$ consisting of all functions satisfying conditions (3), (4). It is easily seen that the functional

$$(y, z) = \int_a^b yL_2z\,dt$$

is the scalar product in A.

Since every scalar product is a symmetric functional we have, by Lemma 3, the following

Theorem 2. If y and z are eigenfunctions of problem (2)–(4) corresponding to eigenvalues λ_1 and λ_2, respectively, $\lambda_1 \neq \lambda_2$, then y and z are orthogonal, i.e. $(y, z) = 0$.

Theorem 3. Every four eigenfunctions of problem (2)–(4) corresponding to the same eigenvalue are linearly dependent.

Proof. Let functions y_j $(j = 1, \ldots, 4)$ be eigenfunctions of problem (2)–(4) corresponding to an eigenvalue λ.

Suppose y_j are linearly independent. Then each solution of the equation is expressed by

$$y(t) = \sum_{j=1}^{n} C_j y_j(t),$$

C_j being arbitrary constants. Therefore, y is a solution of problem (2)–(4) as the linear combination of such solutions. Hence

$$y(a) = \frac{\alpha_2}{\alpha_1} y'(a), \quad y(b) = -\frac{\beta_2}{\beta_1} y'(b)$$

independently of the choice of constants C_j. On the other hand, since y_j are linearly independent, one may choose constants C_j so that (3) does not hold true. This contradiction completes the proof.

4. Theorem 4. Let $\{u_k\}$ be a sequence of eigenfunctions of problems (2)–(4) corresponding to eigenvalues $\{\lambda_k\}$, respectively, If the system $\{u_k\}$ is a complete system in $L^2([a, b])$, then the sequence $\{\lambda_k\}$ contains all the eigenvalues of problem (2)–(4). Furthermore, every eigenfunction of problem (2)–(4) is a finite combination of functions belonging to $\{u_k\}$.

Proof. Take an eigenfunction $u_0, u_0 \notin \{u_k\}$ corresponding to an eigenvalue λ_0. If $\lambda_0 \notin \{\lambda_k\}$, then $(u_0, u_k) = 0$ for $k = 1, 2, \ldots$ Hence and by Parseval identity, $(u_0, u_k) = 0$. Therefore $u_0 \equiv 0$.

Assume now that $\lambda_0 \in \{\lambda_k\}$. Then there exist at most three elements $\lambda_{r_j} \in \{\lambda_k\}$ such that $\lambda_0 = \lambda_{r_1} = \lambda_{r_2} = \lambda_{r_3}$. Indeed suppose $\lambda_0 = \lambda_{r_j}$ for $j = 1, \ldots, 4$, where $\lambda_{r_j} \in \{\lambda_k\}$, $r_i \neq r_j$ for $i \neq j$. Let u_{r_j} $(j = 1, \ldots, 4)$ be eigenfunctions corresponding to λ_{r_j}, respectively. Then, in accordance with Theorem 3, u_{r_j} must be linearly dependent. But this is not possible since every orthogonal system is linearly independent. Put

$$\bar{u}_0(t) = u_0(t) + \sum_{j=1}^{3} a_j u_{r_j}(t),$$

where $a_j = -(u_0, u_{r_j})$. It is easily seen that $(\bar{u}_0, u_j) = 0$ for $j = 1, 2, \ldots$, i.e. $\bar{u}_0$ is orthogonal to every function belonging to $\{u_k\}$. Thus $\bar{u}_0 = 0$. Therefore

$$u_0(t) = \sum_{j=1}^{3} - a_j u_{r_j}(t)$$

The proof is completed.

5. The considerations relating to the problem (2)–(4) we have so far been working-out do not give a method of construction of an eigenfunction complete system and an eigenvalue system corresponding to it which, in accordance with Theorem 4, exhaust all the eigenvalues of problem (2)–(4). We shall now give a construction. Our method will based on some results given in [1].

Denote by B the set of functions belonging to the class $C^k([a,b])$ ($k \geqslant 2$) and satisfying condition (3).

Let $\mathcal{M}$ denote the image of the set B by the operator L_2. It follows from Lemma 2 that there exists an operator K: inverse to L_2, i.e. $K = L_2^{-1}$. Consider the equation

$$L_1 v - \lambda K v = 0 \tag{12}$$

with boundary conditions

$$\alpha_3 v(a) - \alpha_4 v'(a) = 0\,, \qquad \beta_3 v(b) + \beta_4 v'(b) = 0\,, \tag{13}$$

where α_3, β_3, α_4, β_4 are defined exactly as in [3]. It is seen that problem (2)–(4) is equivalent to problem (10), (11) for $k \geqslant 4$.

Theorem 5. The operator K is positive, i.e.

$$\int_a^b \varphi K\varphi\, dt > 0 \quad \text{for } \varphi \in \mathcal{M},\ \varphi \not\equiv 0\,.$$

Proof. Given function $\varphi \in \mathcal{M}$, $\varphi \not\equiv 0$, there exists $u \in B$, $u \not\equiv 0$, such that $\varphi = L_2 u$. Hence

$$\int_a^b \varphi K\varphi\, dt = \int_a^b u L_2 u\, dt = (u, u) > 0\,.$$

Theorem 6. The operator K is symmetric, i.e.

$$\int_a^b \psi K\varphi\, dt = \int_a^b \varphi K\psi\, dt \quad \text{for } \varphi, \psi \in \mathcal{M}\,.$$

Proof. Take functions $\varphi, \psi \in \mathcal{M}$. There exist elements u and v of B such that $\varphi = L_2 u$ and $\psi = L_2 v$. We have

$$\int_a^b \psi K\varphi\, dt = \int_a^b (L_2 v) u\, dt = (u, v) = (v, u) = \int_a^b v L_2 u\, dt = \int_a^b \varphi K\psi\, dt$$

It follows from Lemma 2 and Theorems 1, 2 that functional H: $\mathcal{M} \times \mathcal{M} \to R$ given by

$$H(\varphi, \psi) = \int_a^b \varphi K\psi\, dt \quad \text{for } \varphi, \psi \in \mathcal{M} \tag{14}$$

is the scalar product in $\mathcal{M}$; moreover, if $u = K\varphi$ and $v = K\psi$, then

$$(u, v) = H(\varphi, \psi) \tag{15}$$

5. We shall find eigenvalues and eigenfunctions of problems (12), (13) by means of some variational methods (comp. [1]). To this end we put

$$D(\varphi, \psi) = \int_a^b (p_1 \varphi' \psi' + q_1 \varphi \psi)\, dt + \psi(b) \frac{\beta_3}{\beta_4} \varphi(b) p_1(b) + \psi(a) \frac{\alpha_3}{\alpha_4} \varphi(a) p_1(a) \tag{16}$$

$$D(\varphi, \varphi) = D(\varphi)\,, \qquad H(\varphi, \varphi) = H(\varphi)\,,$$