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On the Approximation Theorem for Boundary Value Problems for Functional Differential Equations

INTRODUCTION

The present note is a continuation of the paper [2], in which the problems of existence and uniqueness of solutions of boundary value problems for functional differential equations were considered. The purpose of this note is to investigate the dependence of solutions of boundary value problems for functional equations on their right-hand sides. The main result of the paper is that if the right-hand sides of functional equations tend to right-hand sides of differential equations (in the sense to be defined later) then the corresponding solutions of boundary value problems for functional equations converge to the solution of the boundary value problem for the ordinary differential equation. This result is a natural extension of A. Lasota's result [1] to the case of ordinary differential and difference equations.

Some ideas in this paper were suggested by [3] but the results obtained here are independent of those of Petryshyn.

1. THE PROBLEM

Let R^m denote a real Euclidean m-space with norm $|\cdot|$. Let $C_{[a,b]}$ $(a < b)$ be a Banach space of all continuous functions $x\colon [a, b] \to R^m$ with uniform norm $\|\cdot\|$.

For $a = b$, $C_{[a,b]}$ is identified with R^m. Moreover for any $b > a$ $C_{a,a]} = R^m$ will always be considered as a subspace of $C_{[a,b]}$ consisting of all constant mappings $[a, b] \to R^m$.

Let $C_s = C_{[a-s,a]}$ $(s \geqslant 0)$. For any $x \in C_{[a-h,b]}$ and $t \in [a, b]$, let $x_t^s \in C_s$ be defined by $x_t^s(\tau) = x_t^s(t+\tau-a)$. (Observe that this definition of x_t coincides with that given in [2]).

By $cf(R^m)$ we denote the space of all non-empty, compact convex subsets of R^m with the Hausdorff distance:

$$d(A, D) = \max(\sup_{q \in D} \delta(q, A), \sup_{q \in A} \delta(q, D)), \quad A, D \in cf(R^m)$$

where

$$\delta(q, A) = \inf\{|q-p| : p \in A\}.$$

Let B denote the Banach space and let $F: B \to cf(R^m)$. The mappings F will be called continuous in B if it is continuous in the Hausdorff topology. The mapping F will be called uper semi continuous in B if for any $u_0 \in B$ conditions $u_n \to u_0$, $v_n \in F(u_n)$, $v_n \to v_0$ as $n \to \infty$ imply $v_0 \in F(u_0)$.

Let $\{f_s\}_{s \in [0,h]}$, $\{F_s\}_{s \in [0,h]}$ be the families of maps $f_s: [a, b] \times C_s \to R^m$, $F_s: [a, b] \times \times C_s \to cf(R^m)$ and let $L: C_{[a,b]} \to R^m$. Consider boundary value problmes

$$x' = f_0(t, x), \quad Lx = r \quad (r \in R^m) \tag{1.1}$$

$$x' = f_s(t, x_t^s), \quad Lx = r \quad (s \in (0, h], r \in R^m). \tag{1.2}$$

In this paper we consider the relationship between the solutions of (1.1) and (1.2) as $s \to 0$.

Together with problems (1.1), (1.2) the homogeneous boundary value problems for the equations with multi-valued right sides will be considered

$$x' \in F_0(t, x), \quad Lx = 0 \quad (r \in R^m), \tag{1.3}$$

$$x' \in F_s(t, x_t^s), \quad Lx = 0 \quad (r \in R^m, s \in (0, h]). \tag{1.4}$$

By a solution $x(t)$ of boundary value problem (1.1) we mean any absolutely continuous function on $[a, b]$ satisfying the conditions $x'(t) = f_0(t, x(t))$, $Lx = r$.

In like manner we define the solutions of boundary value problems (1.2), (1.3), (1.4) (see [2]).

We shall assume in the sequel that f_s, F_s, for $s \in [0, h]$ and L satisfy the assumptions:

(i) $F_s(t, p)$ satisfies the Carathéodory conditions (see [2]), $F_s(t, \lambda p) = \lambda F_s(t, p)$ for $\lambda \in R^1$ and moreover

$$\sup_{\|p\|_s = 1} |F_s(t, p)| \leq \varphi(t)$$

where $\varphi(t)$ is integrable on $[a, b]$.

(Here and below $\|\cdot\|_s$ denotes the norm $|\cdot|$ if $s = 0$ and the norm of C_s in the case $s > 0$);

(ii) $f_s(t, p)$ satisfies the Carathéodory conditions and moreover

$$\lim_{n \to \infty} \frac{1}{n} \int_a^b \sup_{\|p\|_s \leq n} \delta(f_s(t, p), F_s(t, p))\, dt = 0$$

(iii) L is linear and continuous;

(iv) $f_s(t, p)$ is measurable in t for fixed p and, in addition, satisfies the following "generalized Lipschitz conditions"

$$f_s(t, p) - f_s(t, q) \in F_s(t, p-q)$$

and the conditions

$$\int_a^b |f_s(t, 0)|\, dt < \infty .$$

2. A CONVERGENCE OF FAMILIES OF MAPPINGS

Let X be the metric space with metric ϱ. Let $\{G_s\}_{s \in [0,h]}$ be the family of mappings of C_s into X.

Definition 2.1. The family $\{G_s\}_{s \in [0,h]}$ is said to be convergent to G_0 as $s \to 0$ ($G_s \to G_0$ as $s \to 0$) if conditions $\alpha^s \in C_s$, $y^s \in G_s(a^s)$, $\|a^s - u\|_s \to 0$, $\varrho(y^s, v) \to 0$ where $u \in C_0 = R^m$, $v \in X$ imply that $v \in G_0(u)$.

We shall use this definition in the case $X = R^m$ and $X = cf(R^m)$, i.e. in the case of single or multi-valued mappings.

Example 2.1. Let G: $C_h \to cf(R^m)$. For $a \in C_h$ we write

$$[a]_s(\tau) \overset{\mathrm{df}}{=} \begin{cases} \alpha(s), & a-h \leqslant \tau \leqslant a-s \\ \alpha(\tau), & a-s < \tau \leqslant a . \end{cases} \qquad (0 < s \leqslant h)$$

Define the family of mappings $\{G_s\}$ $(0 < s \leqslant h)$ by

$$G_s(a) \overset{\mathrm{df}}{=} G([a]_s) \quad (0 < s \leqslant h) \tag{2.1}$$

and let $G_0 = G|_{R^m}$ — the restriction of G to R^m be identified with the set of constant maps of $[a-h, h]$ into R^m.

It is easy to verify that G is upper semicontinuous (see [2]), then G_s converges to G_0 as $s \to 0$.

Since the single-valued map be considered as a special case of multivalued mappings, the above example has the following counterpart

Example 2.2. Let g: $C_h \to R^m$ be continuous. Then the maps g_s defined by

$$g_s(a) = g([a]_s), \qquad s \in [0, h] \tag{2.2}$$

converge to $g_0 = g|_{R^m}$ as $s \to 0$.

Example 2.3. Suppose that g: $R^m \to R^m$ is continuous. Let g_s be defined by

$$g_s(a) = g\Big(\int_{a-s}^{a} \alpha(\tau)\, d\varphi_s(\tau) \Big)$$

where $\alpha \in C_{[-s,0]}$, the function $\varphi_s(\tau)$ is increasing and, in addition, $\varphi_s(a) - \varphi_s(a-s) = 1$. It is easy to see that g_s is convergent to g in the sense of definition 2.1.

Indeed, let $\alpha^s \in C_s$, $a \in C$, and $\|\alpha^s - a\|_s \to 0$ as $s \to 0$. Then

$$g\Big(\int_{a-s}^{a} \alpha^s(\tau)\, d\varphi_s(\tau)\Big) = g\Big(\int_{a-s}^{a} [(\alpha^s(\tau)-a)+a]\, d\varphi_s(\tau)\Big) = g(\varepsilon(s)+a) \to g(a) \quad \text{as } s \to 0,$$

because

$$\varepsilon(s) = \int_{a-s}^{a} [\alpha^s(\tau)-a]\, d\varphi_s(\tau) \to 0 \quad \text{as } s \to 0.$$

3. APPROXIMATION THEOREM FOR CONTINGENT EQUATIONS

Lemma 3.1. Let the mappings $F_s\colon [a, b] \times C_s \to cf(R^m)$ $(0 \leqslant s \leqslant h)$ satisfy (i) and let for any $t \in [a, b]$, $F_s(t, \cdot) \to F_s(t, \cdot)$ as $s \to 0$ in the sense of Def. 2.1. Let $\alpha^s \in C_s$, $s \in (0, h)$, $x \in C_0 = R^m$ satisfy $\|\alpha^s - x\|_s \to 0$ as $s \to 0$.

Then for every $\varepsilon > 0$ and almost all $t \in [a, b]$ there is an $s_0(t) \to 0$ such that for $0 < s < s_0(t)$

$$F_s(t, \alpha^s) \subset F_\varepsilon(t, x)$$

where

$$F_\varepsilon(t, x) = \{y \in R^m\colon \delta(y, F_0(t, x)) < \varepsilon\}.$$

Proof. We shall show that Lemma 3.1 holds for all t, for which $\varphi(t)$ is bounded. Assume the contrary. Let for some $\varepsilon > 0$ be a sequence $s_k \to 0$ such that $F_{s_k}(t, \alpha^{s_k}) \not\subset F_\varepsilon(t, x)$. Let $y_k \in F_{s_k}(t, \alpha^{s_k}) \setminus F_\varepsilon(t, x)$. By (i), $|y_k|$ are bounded, because $\varphi(t)$ is bounded. Passing, if necessary, to the subsequence we may assume that $\lim\limits_{k\to\infty} y_k = y_0$. We have $\delta(y_0, F_0(t, x)) \geqslant \varepsilon$, i.e. $y_0 \notin F_0(t, x)$, which contradicts $F_{s_k}(t, \alpha^{s_k}) \to F_0(t, x)$ as $s \to 0$.

Theorem 3.1. Let the mappings F_s $(0 \leqslant s \leqslant h$ and L satisfy the conditions (i) and (ii) respectively. In addition, let for any $t \in [a, b]$, $F_s(t, \cdot) \to F_0(t, \cdot)$ in the sense of Definition 2.1. Assume that the problem (1.3) has the unique solution $x = 0$. Then there exists $h_0 > 0$ (depending on L and F_0) such that for $0 < s < h_0$, problem (1.4) has exactly one solution.

Proof. Suppose, if possible, that for each k there exists $0 < s_k < 1/k$ such that for $s = s_k$ (1.4) has a non-trivial solution x^k, i.e.

$$(x^k(t))' \in F_{s_k}(t, x_t^k), \quad Lx^k = 0, \quad x^k(t) \not\equiv 0 \text{ on } [a, b]. \tag{3.1}$$

From the homogeneity of $F_s(t, \alpha)$ and L, we may assume that

$$\|x^k(t)\| = 1. \tag{3.2}$$

From (3.1), (3.2), and (i), it follows that the family $x^k(t)$ is uniformly bounded and equicontinuous on $[a, b]$, hence by Arzela theorem one can choose the subsequence

(3.3) $$x^{\nu_k}(t) \to x_0(t), \text{ uniformly in } [a, b] \text{ as } \nu_k \to 0 .$$

By (3.1), (3.3) and Lemma 3.1

$$\delta(x^{\nu_k}(t), F_0(t, x_0)) \to 0 \quad \text{as } \nu_k \to 0 \text{ a.e. on } [a, b] ,$$

which by Pliś Lemma [4] implies that

$$x_0'(t) \in F(t, x_0) , \qquad Lx = 0 .$$

Obviously $\|x_0\| = 1$ which contradicts the assumption that problem (1.3) has only the trivial solution.

As a corollary of the above results and Theorems 3.1 and 3.2 in [2] we obtain

Theorem 3.2. Let for any $t \in [a, b]$ the family of mappings F_s $(0 \leqslant s \leqslant h)$ be convergent to function F_0 in the sense of Definition 2.1. Suppose problem (1.3) has only the trivial solution. Then:

1° If the mappings F_s, f_s, L satisfy (i), (ii), (iii) then for a sufficiently small s there exists at least one solution of the problem (1.2).

2° If F_s, L, f_s satisfy (i), (ii), (iv) then for a sufficiently small s there exists exactly one solution of problem (1.2).

4. APPROXIMATION THEOREM

Theorem 4.1. Let F_s: $[a, b] \times C_s \to cf(R^m)$, f_s: $[a, b] \times C_s \to R^m$, L: $C_{[a,b]} \to R^m$ satisfy (i), (iii), (iv) for $s \in [0, h]$, Let families $\{F_s\}_{s \in [0,h]}$, $\{f_s\}_{s \in [0,h]}$ be convergent respectively to F_0 and f_0 in the sense of Definition 2.1.

If (1.3) has just a trivial solution, then:

1° there is a unique solution $\mathring{x}$ to (1.1);

2° there is a $h_0 > 0$ such that all $s \in (0, h_0]$, (1.2) have exactly one solution x^s;

3° $\lim\limits_{S \to 0} \|x^s - x\|_s = 0$.

Proof. The first statement of Theorem 4.1 follows from the Lasota Theorem [1]. The second is exactly Theorem 3.1. To prove the last part, let $\mathring{x}(t)$ $(x^s(t))$ be a solution of problem (1.1) (of problem (1.2) for $s < h_0$).

We write

$$u^s(t) = x^s(t) - \mathring{x}(t) .$$

To prove the theorem it suffices to show that $u^s(t) \to 0$ as $s \to 0$,

Obviously

$$(u^s(t))' = f_s(t, x_t^s) - f_0(t, \mathring{x}) = f_s(t, x_t^s) - f_s(t, \mathring{x}_t) + f_s(t, \mathring{x}_t) - f_0(t, \mathring{x}) .$$