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A Difference Method for a Non-linear Integro-differential Equation of the First Order

1. In this paper we shall consider the following integro-differential equation

$$\frac{\partial u}{\partial \xi} = f\left(\xi, x, u, \frac{\partial u}{\partial x}, \int_0^d u(\xi, x)dx\right). \tag{1.1}$$

We construct the corresponding difference equation and we prove the convergence of the difference method in the p-dimensional case.

The paper gives generalizations of results obtained by Z. Kowalski [2] for a differential equation.

2. Let us denote by m the sequence of p natural numbers

$$m = (m_1, \ldots, m_p) \tag{2.1}$$

and let

$$M = (\mu, m), \tag{2.2}$$

where μ is a natural number.

We shall consider the points x^m of the real p-dimensional space R^p with coordinates

$$x^m = (x_1^{m_1}, \ldots, x_p^{m_p}) \tag{2.3}$$

and also the nodal points

$$(\xi^\mu, x^m) \in R^{p+1} \tag{2.4}$$

ξ^μ and x^m being defined by

$$\xi^\mu = \mu k, \quad x_j^v = v \cdot h, \tag{2.5}$$

$$\mu = 0, 1, \ldots, \quad v = 0, 1, \ldots, \quad j = 1, \ldots, p,$$

$$0 < h = \text{const}, \quad 0 < k = \text{const}$$

for

$$(\xi^\mu, x_1^{m_1}, \ldots, x_p^{m_p}) \in E,$$

where

$$E\colon 0 \leqslant \xi \leqslant d, \quad 0 \leqslant x_j \leqslant d, \quad d > 0, \quad j = 1, \dots, p. \tag{2.6}$$

We define the following sequences of indices

$$\omega(M) = (u+1, m), \quad j(M) = (\mu, j(m)) \tag{2.7}$$

where

$$j(m) = (m_1, \dots, m_{j-1}, m_j - 1, m_{j+1}, \dots, m_p), \quad j = 1, \dots, p,$$

Suppose that to each M there corresponds a number v^M. Index n_h being chosen so that

$$n_h \cdot h \leqslant d \quad \text{and } (n_h+1)\cdot h > d,$$

and putting $\sigma = d - n_h \cdot h$,
we introduce the following differences and the sum

$$v^{M\sim} = \frac{1}{k}(v^{\omega[M]} - v^M),$$

$$v^{M_j} = \frac{1}{h}(v^M - v^{j[M]}), \quad v^{M\Delta} = (v^{M_1}, \dots, v^{M_p}), \tag{2.8}$$

$$v^{\mu\Box} = \sum_{M \in \hat{E}^\alpha} v^M dM,$$

where

$$\hat{E}^\mu = \{M = (\mu, m_1, \dots, m_p)\colon 0 \leqslant m_i \leqslant n_h, \ (i = 1, \dots, p)$$

and

$$dM = \sigma^k h^{p-k} \tag{2.9}$$

for $M = (\mu, m_1, \dots, m_p)$ such that $m_i = n_h$ for k indices i $(0 \leqslant k \leqslant p)$, $m_i < n_h$ for the remaining $p-k$ indices i. We have obviously

$$\sum_{M \in \hat{E}^\mu} dM = d^p. \tag{2.10}$$

3. Throughout the rest of the paper we shall use the following assumptions H.
Assumptions H.

1° Assume that the scalar function $f(\xi, x, u, q, \delta)$, where

$$x = (x_1, \dots, x_p), \quad q = (q_1, \dots, q_p)$$

is of the class C^1 in the region

$$D\colon \begin{cases} 0 \leqslant \xi \leqslant d, \quad 0 \leqslant x_j \leqslant d, -\infty < u < +\infty, \\ -\infty < q_j < +\infty, -\infty < \delta < ++\infty, \quad j = 1, \dots, p. \end{cases} \tag{3.1}$$

2° The derivatives f_u, f_{q_j}, f_δ fulfil conditions

$$|f_u| \leqslant L, \quad f_{q_j} \leqslant 0, \quad |f_\delta| \leqslant K, \quad \sum_j f_{q_j}^2 \neq 0 \tag{3.2}$$

the mesh size h and k being defined so as to obtain

$$\sum_{j=1}^{p} f_{q_j} + \frac{h}{k} \geqslant 0 \quad \text{for} \quad (\xi, x, u, q, v) \in D. \tag{3.3}$$

3° The scalar function $u(\xi, x)$ of the class C^1 satisfies the integro-differential equation

$$\frac{\partial u}{\partial \xi} = f\left(\xi, x, u, \frac{\partial u}{\partial x}, \int_0^d u(\xi, x)\,dx\right) \tag{3.4}$$

for $(\xi, x) \in E$, where

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right),$$

and the boundary conditions

$$\begin{cases} u(0, x) = \varphi_0(x), \\ u(\xi, x) = \varphi_j(x) \quad \text{for } (\xi, x) \in E, \quad x_j = 0, \quad j = 1, \ldots, p. \end{cases} \tag{3.5}$$

4. We accept the following boundary conditions for the numbers v^M.

$$v^M = \begin{cases} \varphi_0(x^m) & \text{for } M = (0, m), \\ \varphi_j(\xi^\mu, x_1^{m_1}, \ldots, x_j^0, \ldots, x_p^{m_p}) & \text{for} \end{cases} \tag{4.1}$$

$$\mu = 0, 1, \ldots, \quad M = (\mu, m_1, \ldots, 0, \ldots, m_p),$$
$$j = 1, \ldots, p,$$

the values v^M for the remaining M being defined successively with the aid of the difference equation

$$v^{M\sim} = f(\xi^\mu, x^m, v^M, v^{M\Delta}, v^{\mu\Box}). \tag{4.2}$$

We denote by u^M the value of solution $u(\xi, x)$ of equation (3.4) at the nodal point (2.4) and we define the corresponding differences as in the case of numbers v^M. The boundary conditions (3.5) imply the boundary conditions for u^M

$$u^M = \begin{cases} \varphi_0^{(x^m)} & \text{for } M = (0, m) \\ \varphi_j(\xi^\mu, x_1^{m_1}, \ldots, x_j^0, \ldots, x_p^{m_p}), \end{cases} \tag{4.3}$$

$$\text{for } \mu = 0, 1, \ldots, \quad j = 1, \ldots, p, \quad M = (\mu, m_1, \ldots, 0, \ldots, m_p).$$

The numbers u^M satisfy the difference equation

$$u^M = f(\xi^\mu, x^m, u^M, u^{M\Delta}, u^{\mu\Box}) + \eta^M \tag{4.4}$$

for all nodal points in E,

where $\max_M |\eta^M| \to 0$ as $h \to 0$, since $h \to 0$ implies by (3.2), (3.3) $k \to 0$ and consequently $u^{M\sim}, u^{M\Delta}, u^{\mu\Box}$ tend to $\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial x}, \int_0^d u(\xi, x)\,dx$ respectively.

5. Lemma 1. Suppose that the numbers R^μ satisfy the difference inequalities

(5.1) $$R^{\mu\sim} \leqslant L_1 R^\mu + \varepsilon \qquad \mu = 0, 1, \ldots,$$

and the initial condition $R^0 = 0$, where

$$R^{\mu\sim} = \frac{1}{H}(R^{\mu+1} - R^\mu), \qquad 0 < H = \text{const},$$
$$0 < L_1 = \text{const},$$
$$0 < \varepsilon = \text{const}.$$

Under these assumptions

(5.2) $$R^\mu \leqslant \frac{\varepsilon}{L_1}(e^{L_1 H \mu} - 1).$$

This Lemma is due to Z. Kowalski [2].

6. Lemma 2. Suppose that the assumptions H are fulfilled and the values u^M and v^M satisfy (4.3), (4.4) and (4.1), (4.2) respectively at the nodal points in the region E. Let us write

(6.1) $$r^M = u^M - v^M$$

(6.2) $$s^\mu = \max_m (r^{\mu,m}), \qquad z^\mu = \min_m (r^{\mu,m}) \qquad \mu = 0, 1, \ldots$$

By (4.1), (4.3) we have obviously $s^\mu \geqslant 0$, $z^\mu \leqslant 0$. Under these assumptions the numbers s^μ and z^μ satisfy respectively the inequalities

(6.3) $$s^{\mu\sim} \leqslant L \cdot |r^{\mu,a}| + K \cdot \sum_{M \in \hat{E}^\mu} |r^M| dM + \varepsilon(h),$$
$$z^{\mu\sim} \geqslant -L|r^{\mu,c}| - K \sum_{M \in \hat{E}^\mu} |r^M| dM - \varepsilon(h),$$

where $0 \leqslant \varepsilon(h) \to 0$ as $h \to 0$, and (μ, a), (μ, c) denote certain nodal points.

Proof. There exist $a = (a_1, \ldots, a_p)$ and $b = (b_1, \ldots, b_p)$ such that

(6.4) $$s^{\mu+1} = \max_m r^{\mu+1,m} = r^{\mu+1,a},$$

(6.5) $$s^\mu = \max_m r^{\mu,m} = r^{\mu,b},$$

whence

(6.6) $$s^{\mu\sim} = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,b}) = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}).$$

Because of (6.1) we have

(6.7) $$\frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = \frac{1}{k}(u^{\mu+1,a} - v^{\mu+1,a}) -$$
$$-\frac{1}{k}(u^{\mu,a} - v^{\mu,a}) = \frac{1}{k}(u^{\mu+1,a} - u^{\mu,a}) - \frac{1}{k}(v^{\mu+1,a} - v^{\mu,a}).$$

If for some j, $1 \leqslant j \leqslant p$, we have $a_j = 0$, inequalities (6.3) hold true, since by the boundary conditions $r^{\mu+1,a} = r^{\mu,a} = 0$ and $r^{\mu,b} \geqslant 0$ by (6.5).

However, for $a_j \geqslant 1$, $j = 1, \ldots, p$, we have

$$(6.8)\quad \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) = f(\xi^\mu, x^a, u^{\mu,a}, u^{(\mu,a)\Delta}, u^{\mu\Box}) + \eta^{\mu,a} - f(\xi^\mu, x^a, v^{\mu,a}, v^{(\mu,a)\Delta}, v^{\mu\Box}).$$

We apply the mean value theorem to the right hand member of (6.8) and putting $A = (\mu, a)$ we get

$$(6.9)\quad s^{\mu\sim} = \frac{1}{k}(r^{\mu+1,a} - r^{\mu,a}) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) = \eta^{\mu,a} + f_u(\sim)(u^{\mu,a} - v^{\mu,a}) +$$
$$+ \sum_{j=1}^{p} f_{q_j}(\sim)(u^{A_j} - v^{A_j}) + f_\delta(\sim)(u^{\mu\Box} - v^{\mu\Box}) +$$
$$+ \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) = \eta^{\mu,a} + f_u(\sim) r^{\mu,a} +$$
$$+ \frac{1}{h} \sum_{j=1}^{p} f_{q_j}(\sim)(u^{\mu,a} - v^{\mu,a} + v^{\mu,j(a)} - u^{\mu,j(a)}) +$$
$$+ fy(\sim) \sum_{M \in \hat{E}^\mu} \Big(u^M dM - \sum_{M \in \hat{E}^\mu} v^M dM) + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}) \Big) =$$
$$= \eta^{\mu,a} + f_u(\sim) r^{\mu,a} + \frac{1}{h} \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,j(a)}) +$$
$$+ f_\delta(\sim) \sum_{M \in \hat{E}^\mu} r^M dM + \frac{1}{k}(r^{\mu,a} - r^{\mu,b}).$$

Now we majorize the right hand member of (6.9). From (6.5) follows

$$(6.10)\quad r^{\mu,j(a)} \leqslant r^{\mu,b},$$

whence

$$(6.11)\quad r^{\mu,a} - r^{\mu,j(a)} \geqslant r^{\mu,a} - r^{\mu,b}$$

and

$$(6.12)\quad \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,j(a)}) \leqslant \sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a} - r^{\mu,b}),$$

since $f_{q_j}(\sim) \leqslant 0$ by assumptions H.

It follows by (6.9), (6,12) that

$$(6.13)\qquad s^{\mu\sim} \leqslant \eta^{\mu a}+f_u(\sim)r^{\mu,a}+\frac{1}{h}\sum_{j=1}^{p} f_{q_j}(\sim)(r^{\mu,a}-r^{\mu,b})+$$

$$+f_\delta(\sim)\sum_{M\in\hat{E}^\mu} r^M dM+\frac{1}{k}(r^{\mu,a}-r^{\mu,b})$$

$$=\eta^{\mu,a}+f_u(\sim)r^{\mu,a}+f_\delta(\sim)\sum_{M\in\hat{E}^\mu} rM_{dM}+$$

$$+\frac{1}{h}(r^{\mu,a}-r^{\mu,b})\left[\sum_{j=1}^{p} f_{q_j}(\sim)+\frac{h}{k}\right].$$

The last term of the right hand member of (6.13) is non-positive by (3.3) and the inequality $r^{\mu,a}-r^{\mu,b}\leqslant 0$. Hence by (3.2) we get the first inequality (6.3)

$$s^{\mu\sim}\leqslant\varepsilon(h)+L|r^{\mu,a}|+K\sum_{M\in\hat{E}^\mu}|r^M|\,dM\,,$$

where $\varepsilon(h)=\max_M|\eta^M|$.

The second inequality (6.3) for z^μ can be proved in a similar way.

This completes the proof of Lemma 2.

7. The following Lemma [3] is obvious

Lemma 3. The quantities s^μ, z^μ being defined by (6.2) we have

$$(7.1)\qquad [\max_m|r^{\mu,m}|]^\sim\leqslant\max(s^{\mu\sim},\,-z^{\mu\sim})\,.$$

8. Lemma 4. Suppose that the assumptions H are fulfilled and let us denote

$$(8.1)\qquad R^\mu=\max_m|r^{\mu,m}|\,.$$

Under these assumptions R^μ satisfies the difference inequality

$$(8.2)\qquad R^{\mu\sim}\leqslant(L+K\cdot d^p)R^\mu+\varepsilon(h)$$

and the initial condition $R^0=0$.

Proof: $R^0=0$ since $r^M=u^M-v^M=0$ at the boundary nodal points.

From (8.1) and Lemma 3 it follows that

$$(8.3)\qquad R^{\mu\sim}=(\max_m|r^{\mu,m}|)^\sim\leqslant\max_m(s^{\mu\sim},\,-z^{\mu\sim})\,.$$

But s^μ and z^μ satisfy the inequalities (6.3), hence

$$(8.4)\qquad R^{\mu\sim}\leqslant\max\Big(L|r^{\mu,a}|+K\sum_{M\in\hat{E}^\mu}|r^M|\,dM+\varepsilon(h)\,,$$

$$L|r^{\mu,c}|+K\sum_{M\in\hat{E}^\mu}|r^M|\,dM+\varepsilon(h)\Big)\,.$$