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On a Difference Method for Non-linear Parabolic Equations with Mixed Derivatives

1. In this paper we present a difference method for the non-linear parabolic equation with mixed derivatives

$$\frac{\partial u}{\partial t}=f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial x}\right), \tag{1.1}$$

where $x=(x_1, \ldots, x_p)$, $u=u(t, x)$,

$$\frac{\partial u}{\partial x}=\left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_p}\right), \quad \frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_p^2}\right),$$

$$\frac{\partial^2 u}{\partial x \partial x}=\left(\frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_p}, \frac{\partial^2 u}{\partial x_2 \partial x_3}, \ldots, \frac{\partial^2 u}{\partial x_{p-1} \partial x_p}\right)$$

with initial and boundary conditions

$$\begin{cases} u(0, x)=\varphi(x) \\ \dfrac{\partial u}{\partial x_j}=\varphi_j(t, x_j') \quad \text{for } x_j=0, \quad j=1, \ldots, p, \\ \dfrac{\partial u}{\partial x_j}=\psi_j(t, x_j'') \quad \text{for } x_j=a, \quad j=1, \ldots, p, \end{cases} \tag{1.2}$$

where $x_j'=(x_1, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_p)$, $x_j''=(x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_p)$.

Assuming conditions (1.2) with

$$\varphi_j(t, x_j')=\psi_j(t, x_j'')=0 \tag{1.3}$$

for a non-linear parabolic equation without mixed derivatives Z. Kowalski [2] solved the problem by proving the convergence of a difference method. On the other hand for the equation (1.1) and the conditions

$$\begin{cases} u(0,x) = \varphi(x) & \\ u(t,x_j') = \varphi_j(t,x_j') & \text{for } x_j = 0, \quad j = 1, \ldots, p \\ u(t,x_j'') = \psi_j(t,x_j'') & \text{for } x_j = a, \quad j = 1, \ldots, p \end{cases} \tag{1.4}$$

a difference method has been introduced and investigated by A. Fitzke [1].

In the present paper we propose a difference method, similar to that in the paper [1], for solving the equation (1.1) with conditions (1.2). The proof of the convergence is based on our previous results [3] concerning the convergence of various difference methods. An error estimation will also be given.

2. Let us denote

$$D_T = \{(t,x)\colon 0 \leqslant t \leqslant T, \quad 0 \leqslant x_j \leqslant a, \quad j = 1, \ldots, p\} \tag{2.1}$$

In the set D_T we introduce a net with the time-step $k = T/N_1$ and the space-step $h = a/N$, where N_1 and N are positive integers. We denote by $M = (\mu, m)$, where $m = m_1, \ldots, m_p$, the nodal point (t^μ, x^m), with $t^\mu = \mu k$, $x^m = mh$. We shall also use the notation i and j for p-dimensional indices with all coordinates equal to zero except the i-th, resp. the j-th, which equals 1. Here we make the following convention: whenever the i and j are written as upper indices they are to be understood as p-dimensional.

Consider the following two sets Z and Z^+

$$Z = \{(\mu, m)\colon 0 \leqslant \mu \leqslant N_1,\ 0 \leqslant m_j \leqslant N,\ j = 1, \ldots, p\} \tag{2.2}$$

$$Z^+ = \{(\mu, m)\colon 0 \leqslant \mu \leqslant N_1,\ 0 \leqslant m_j \leqslant N,\ j = 1, \ldots, p, \text{ except of at most two values } i \text{ and } j \text{ for which } -1 \leqslant m_i \leqslant N+1,\ -1 \leqslant m_j \leqslant N+1\}. \tag{2.3}$$

We introduce the set Z^+ since we shall consider the difference quotients only at the points of the set Z, but in expressions for difference quotients the points of Z^+ may occur.

The values of a function v at the nodal points M we denote by v^M. Following [1] we define the terms:

$$\delta v^{M_j} = \frac{v^{\mu m+j} - v^{\mu m-j}}{2h} \tag{2.4}$$

$$\delta v^M = (\delta v^{M_1}, \ldots, \delta v^{M_p}) \tag{2.5}$$

$$\sigma v^{M_{jj}} = \frac{1}{h^2(p-1)} \sum_{i \neq j} \big(\alpha(v^{\mu m+i+j} - 2v^{\mu m+i} + v^{\mu m+i-j}) + \beta(v^{\mu m+j} - 2v^{\mu m} + v^{\mu m-j}) + \alpha(v^{\mu m-i+j} - 2v^{\mu m-i} + v^{\mu m-i-j})\big) \tag{2.6}$$

$$\sigma v^M = (\sigma v^{M_{11}}, \ldots, \sigma v^{M_{pp}}) \tag{2.7}$$

$$\varrho v^{M_{ij}} = \frac{1}{4h^2} (v^{\mu m+i+j} + v^{\mu m-i-j} - v^{\mu m+i-j} - v^{\mu m-i+j}) \tag{2.8}$$

$$\varrho v^M = (\varrho v^{M_{12}}, \ldots, \varrho v^{M_{1p}}, \varrho v^{M_{23}}, \ldots, \varrho v^{M_{p-1p}}) \tag{2.9}$$

where i and j on the right-hand sides are the above defined p-dimensional indices and α and β are the positive constants satisfying the condition $2\alpha+\beta=1$.

3. We denote by E the set

$$E=\{(t,x,u,p,q,w)\colon (t,x)\in D_T,\ -\infty<u<+\infty,\ -\infty<p_i<+\infty,\\ -\infty<q_i<+\infty,\ -\infty<w_{ij}<+\infty,\ i=1,\dots,p,\ j=1,\dots,p,\ i<j\}$$

Assumptions A:

(i) the function $f(t,x,u,p,q,w)$ is of class C^1 in the set E,

(ii) there are such positive constants α,β, that $2\alpha+\beta=1$ and the steps k and h may be so chosen that the inequalities

$$\left|\frac{\partial f}{\partial p_i}\right|<\frac{2}{2h}\left(\beta\frac{\partial f}{\partial q_i}-\frac{2\alpha}{p-1}\sum_{j\neq i}\frac{\partial f}{\partial q_j}\right) \tag{3.1}$$

$$\left|\frac{\partial f}{\partial w_{ij}}\right|<\frac{4\alpha}{p-1}\left(\frac{\partial f}{\partial q_i}+\frac{\partial f}{\partial q_j}\right) \tag{3.2}$$

$$1+k\frac{\partial f}{\partial u}-\frac{2\beta k}{h^2}\sum_{i=1}^{p}\frac{\partial f}{\partial q_i}>0 \tag{3.3}$$

hold in the set E,

(iii) the function $u(t,x)$ is of class C^2 in the set D_T and satisfies the equation (1.1) with the conditions (1.2).

4. Now we shall define the values of the solution at the nodal points of $Z^+\setminus Z$

$$\begin{aligned}
u^{\mu m+i}&=u^{\mu m-i}+2h\psi_i(t^\mu,x^m) && \text{for } m_i=N\\
u^{\mu m-i}&=u^{\mu m+i}-2h\varphi_i(t^\mu,x^m) && \text{for } m_i=0\\
u^{\mu m+i+j}&=u^{\mu m-i-j}+2h\big(\psi_i(t^\mu,x^m)+\psi_j(t^\mu,x^m)\big) && \text{for } m_i=m_j=N\\
u^{\mu m-i-j}&=u^{\mu m+i+j}-2h\big(\varphi_i(t^\mu,x^m)+\varphi_j(t^\mu,x^m)\big) && \text{for } m_i=m_j=0\\
u^{\mu m-i+j}&=u^{\mu m+i-j}-2h\big(\varphi_i(t^\mu,x^m)-\psi_j(t^\mu,x^m)\big) && \text{for } m_i=0,\ m_j=N
\end{aligned} \tag{4.1}$$

The u^M just defined satisfies the conditions

$$\begin{aligned}
u^M&=\varphi(x^m) && \text{for } M=(0,m)\in Z\\
\delta u^{M_j}&=\varphi_j(t^\mu,x^m) && \text{for } m_j=0,\ M\in Z,\ j=1,\dots,p\\
\delta u^{M_j}&=\psi_j(t^\mu,x^m) && \text{for } m_j=N,\ M\in Z,\ j=1,\dots,p
\end{aligned} \tag{4.2}$$

The first condition follows from assumptions A, the other two from (4.1).

5. **Lemma 1.** If the assumptions A are fulfilled, then we have

$$\frac{u^{\mu+1m}-u^{\mu m}}{k}=f(t^\mu,x^m,u^M,\delta u^M,\sigma u^M,\varrho u^M)+\eta^M \tag{5.1}$$

$$\text{for } 0\leqslant\mu\leqslant N_1-1,\ 0\leqslant m_j\leqslant N,\ j=1,\dots,p,$$

$$\lim_{h\to 0}\max_M|\eta^M|=0. \tag{5.2}$$

Proof. Since the function u is of class C^2 in D_T, then (5.1) and (5.2) are satisfied at any inner nodal point M for which $1 \leqslant m_j \leqslant N-1$. We shall show that the equation (5.1) is also satisfied for the boundary nodal points. By (4.2) it is clear that $\delta u^{Mj} \to \frac{\partial u}{\partial x_j}$ as $h \to 0$, thus it remains to prove that for boundary nodal points we have

1° $$\sigma u^{Mjj} \to \frac{\partial^2 u}{\partial x_j^2} \quad \text{as } h \to 0,$$

2° $$\varrho u^{M_{ij}} \to \frac{\partial^2 u}{\partial x_i \partial x_j} \quad \text{as } h \to 0.$$

Suppose that $m_j = 0$ i.e. the nodal point is on the hyperplane $x_j = 0$. Consider the three groups of indices:

$$I_1 = \{i\colon (\mu, m+i) \in Z \text{ and } (\mu, m-i) \in Z\}$$
$$I_2 = \{i\colon m_i = 0\}$$
$$I_3 = \{i\colon m_i = N\}.$$

Using (2.6) and (4.1) we have

$$\sigma u^{Mjj} = \frac{2\alpha}{(p-1)h^2} \sum_{i \in I_1} u^{\mu m+i+j} - u^{\mu m+i} + u^{\mu m-i+j} - u^{\mu m-i} - h\big(\varphi_j(t^\mu, x^{m+i}) + \varphi_j(t^\mu, x^{m-i})\big) +$$
$$+ \sum_{i \in I_2} 2(u^{\mu m+i+j} - u^{\mu m+i}) + h(\varphi_j(t^\mu, x^m) + \varphi_i(t^\mu, x^m) - \varphi_i(t^\mu, x^{m+j}) - \varphi_j(t^\mu, x^{m+i}) +$$
$$+ \sum_{i \in I_3} 2(u^{\mu m-i+j} - u^{\mu m-i}) + h\big(\psi_i(t^\mu, x^{m+j}) + \varphi_j(t^\mu, x^{m-i}) - \varphi_j(t^\mu, x^m) - \psi_i(t^\mu, x^m)\big) +$$
$$+ \frac{2\beta}{h^2}\big((u^{\mu m+j} - u^{\mu m} - h\varphi_j(t^\mu, x^m)\big).$$

With the aid of Taylor's expansion at the point (t^μ, x^m) we obtain:
for $i \in I_1$:

$$u^{\mu m+i+j} - u^{\mu m+i} + u^{\mu m-i+j} - u^{\mu m-i} - h\big(\varphi_j(t^\mu, x^{m+i}) + \varphi_j(t^\mu, x^m)\big) =$$
$$= h^2 \frac{\partial^2 u}{\partial x_j^2} + 2h \frac{\partial u}{\partial x_j} - h\big(\varphi_j(t^\mu, x^{m+i}) + \varphi_j(t^\mu, x^{m-i})\big) + 0(h^2),$$

for $i \in I_2$:

$$2(u^{\mu m+i+j} - u^{\mu m+i}) + h\big(\varphi_j(t^\mu, x^m) - \varphi_j(t^\mu, x^{m+i}) + \varphi_i(t^\mu, x^m)\big) -$$
$$- \varphi_i(t^\mu, x^{m+j}) = h^2 \frac{\partial^2 u}{\partial x_j^2} + 2 \frac{\partial^2 u}{\partial x_i \partial x_j} + h\big(\varphi_j(t^\mu, x^m) - \varphi_j(t^\mu, x^{m+i}) +$$
$$+ \varphi_i(t^\mu, x^m) - \varphi_i(t^\mu, x^{m+j})\big) + 0(h^2),$$

for $i \in I_3$:

$$2(u^{\mu m-i+j}-u^{\mu m-i})+h\big(\psi_i(t^\mu, x^{m+j})-\psi_i(t^\mu, x^m)+\varphi_j(t^\mu, x^m)-$$
$$-\varphi_j(t^\mu, x^{m-i})\big)=h^2\frac{\partial^2 u}{\partial x_j^2}-2\frac{\partial^2 u}{\partial x_i \partial x_j}+h\big(\psi_i(t^\mu, x^{m+j})-$$
$$-\psi_i(t^\mu, x^m)+\varphi_j(t^\mu, x^m)-\varphi_j(t^\mu, x^{m-i})\big)+0(h^2)$$

and finally

$$2\big(u^{\mu m+j}-u^{\mu m}-h\varphi_j(t^\mu, x^m)\big)=h^2\frac{\partial^2 u}{\partial x_j^2}+0(h^2)$$

Replacing $\varphi_i, \varphi_j, \psi_i$ by suitable derivatives from (1.2) yields

$$\sigma u^{M_{jj}}=\frac{\partial^2 u}{\partial x_j^2}+0(1) \text{ as } h \to 0$$

that completes the proof of 1° if $m_j = 0$. A similar result can be obtained if we put $m_j = N$.

Now we shall prove 2°. Putting $m_i = N$ and using (2.8) we get for $1 \leqslant m_j \leqslant N-1$ the formula

$$\varrho u^{M_{ij}}=\frac{1}{4h^2}\big(2h\psi_i(t^\mu, x^{m+j})-2h\psi_i(t^\mu, x^{m-j})\big)$$

Hence

$$\varrho u^{M_{ij}}=\frac{\partial^2 u}{\partial x_i \partial x_j}+0(1) \qquad \text{as } h \to 0$$

We can obtain similar results by putting $m_j = 0$ or $m_j = N$. When $m_i = 0$ the proof is similar.

This completes the proof of Lemma 1.

6. Denote

$$|n|=\sum_{s=1}^{p}|n_s|\,, \tag{6.1}$$

where n_s are integers and $n = (n_1, \ldots, n_p)$.

Consider the difference equation with the unknown function $r^{\mu m}$

$$\frac{r^{\mu+1 m}-r^{\mu m}}{k}=L^{\mu m} r^m+\frac{1}{h}\Big(P^0 r^{\mu m}+\sum_{0<|u|\leqslant 2} P^n r^{\mu m+n}\Big)+\eta^{\mu m} \tag{6.2}$$

and put

$$L=\max\{L^{\mu m}\colon\ \mu, m \in Z\} \tag{6.3}$$

$$\varepsilon^\mu=\max_{\nu \leqslant \mu}\max_{m}|\eta^{\nu m}| \qquad (\nu, m) \in Z \tag{6.4}$$

The following Lemma is a particular case of Theorem 1 in paper [3]:

Lemma 2. If

$$P^n \geqslant 0 \text{ for } 0 < |n| \leqslant 2, \quad P^0 + \sum_{0<|n|\leqslant 2} P^n = 0,$$

$$1 + kL^{\mu m} + \frac{k}{h} P^0 \geqslant 0$$

then the solution of (6.2) satisfies the inequality

$$|r^{\mu m}| \leqslant \frac{\varepsilon^\mu}{L}\left(e^{kL\mu} - 1\right)$$

in the set Z.

7. We are going to describe the approximate solution v^M. We put

$$v^{0m} = \varphi(x^m), \quad \delta v^{M_j} = \varphi_j(t^\mu, x^m) \quad \text{for } m_j = 0,\ M \in Z, \tag{7.1}$$

$$\delta v^{M_j} = \psi_j(t^\mu, x^m) \quad \text{for } m_j = N,\ M \in Z, \quad j = 1, \ldots, p.$$

For the nodal points $M \in Z$ we calculate the v^M successively from the equation

$$\frac{v^{\mu+1m} - v^{\mu m}}{k} = f(t^\mu, x^m, v^M, \delta v^M, \sigma v^M, \varrho v^M) \tag{7.2}$$

and for $M \in Z^+ \setminus Z$ the v^M are defined by the right hand sides of (4.1) with u replaced by v.

It is easily verified that the function v is well-defined for any nodal point M.

8. We denote

$$r^M = u^M - v^M \tag{8.1}$$

Now we are ready to prove the following

Theorem. If the assumptions A are fulfilled for the equation (1.1), the accurate solution u^M satisfies (4.1), (4.2), (5.1), the approximate solution v^M is defined by (7.1), (7.2) and (4.1), the error r^M is defined by (8.1) and the function ε^μ by (6.4), then

1° r^M satisfies the inequality (6.5),

2° the difference method (7.2) is convergent i.e. for each $T > 0$

$$\lim_{h\to 0} r^M = 0$$

uniformly in the set D_T.

Proof. Using the mean-value theorem we obtain as in [1]:

$$\frac{r^{\mu+1m} - r^{\mu m}}{k} = \frac{\partial f}{\partial u} r^{\mu m} + \frac{1}{2h}\sum_i \left(\frac{\partial f}{\partial p_i} + \frac{2\beta}{h}\frac{\partial f}{\partial q_i} - \frac{4\alpha}{(p-1)h}\sum_{j\neq i}\frac{\partial f}{\partial q_j}\right) r^{\mu m+i} +$$

$$+ \frac{1}{2h}\sum_i \left(-\frac{\partial f}{\partial p_i} + \frac{2\beta}{h}\frac{\partial f}{\partial q_i} - \frac{4\alpha}{(p-1)h}\sum_{j\neq i}\frac{\partial f}{\partial q_j}\right) r^{\mu m-i} +$$