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Krystyna Lewkow-Jabłońska

On an Integral Inequality

In [1] J. Hale and K. Meyer have considered the following integral inequality

$$u(t) \leqq a+\beta\Big[\int_0^t u^2(s)\,ds\Big]^{1/2} \tag{1}$$

where a and β are given positive real numbers. They have proved that any continuous function $u(t)$ from $[0, \infty)$ to the set R of real numbers satisfying (1) satisfies the inequality

$$u(t) \leqq ak\exp\frac{\beta^2 t}{2}$$

with a constant k independent of the function $u(t)$.

The purpose of this note is to give the precise value of the constant k. Namely, we shall prove the following.

Theorem. For any real continuous function $u(t)$ in the interval $[0, \infty)$ satisfying inequality (1) we have

$$u(t) \leqq ae\exp\frac{\beta^2 t}{2} \tag{2}$$

The constant e in this inequality is the best possible.

Proof. First we shall prove that for any function $u(t)$ satisfying (1) we have the inequality

$$u(t) \leqq U(t) \tag{3}$$

where $U(t)$ is the solution of the integral equation

$$U(t) = a+\beta\Big[\int_0^t U^2(s)\,ds\Big]^{1/2} \tag{4}$$

The solution $U(t)$ of (4) exists in the interval $[0, \infty)$ and is unique, since (4) is equivalent, as we shall see below, to a Cauchy initial value problem for a simple differential equation.

To prove (3) let us consider the sequence $u_1(t), u_2(t), \dots$ of functions defined by the recurrence formula

$$u_1(t) = u(t)\,,$$

$$u_n(t) = a+\beta\Big[\int_0^t u_{n-1}^2(s)\,ds\Big]^{1/2}, \qquad (n = 2, 3, \dots)$$

It is easily seen that the functions $u_2, u_3, \dots$ are increasing and that the sequence $u_1, u_2, \dots$ is increasing, and in virtue of the Hale–Meyer inequality are uniformly bounded in every finite interval.

Moreover,

$$\lim_{n\to\infty} u_n(t)$$

exists and satisfies (4).

Thus, in order to prove our theorem, we shall consider the equation (4), proving the following

Proposition. If a continuous function U: $[0, \infty)\to\mathbf{R}(-\infty, \infty)$ fulfils (4), then

(5) U is strictly increasing,

$$\frac{\beta^2 t}{2} \leqq \ln\frac{U(t)}{a} \leqq \frac{\beta^2 t}{2}+1 \tag{6}$$

$$\lim U(t) = \infty, \quad \text{for } t\to\infty \tag{7}$$

$$\lim_{t\to\infty}\frac{U(t)}{a}\exp\left(-\frac{\beta^2 t}{2}\right) = e \tag{8}$$

Proof of the proposition. The proof of (5) is trivial, since β is positive. To prove (6) it suffices to consider the form of (4). Namely, from (4) we have

$$[U(t)-a]^2 = \beta^2\int_0^t U^2(s)\,ds$$

Moreover the form of (4) implies that the solution of this equation is at least of class C^1 in the interval $[0, \infty)$. Differentiating the last equality, we get

$$2[U(t)-a]\,U'(t) = \beta^2 U^2(t)$$

Hence

$$\frac{[U(t)-a]\,U'(t)}{U^2(t)} = \frac{\beta^2}{2}$$

Integrating, we get

$$\int_a^{U(t)}\frac{U-a}{U^2}\,dU = \frac{\beta^2 t}{2}$$

Consequently,

$$\ln U(t) - \ln \alpha + \frac{a}{U(t)} - 1 = \frac{\beta^2 t}{2} \tag{9}$$

From (4) and (5) we have

$$0 \leqq \frac{a}{U(t)} < 1 \tag{10}$$

Hence, from (9) and (10) we get inequality (6). The proof of (7) is trivial. It follows from (6). To prove (8) it suffices to consider the form (9) again. Namely, from (9) we have

$$\frac{U(t)}{a} \exp \frac{-\beta^2 t}{2} = \exp\left(\frac{-a}{U(t)} + 1\right) \tag{11}$$

Hence, from (7) and (11) we get

$$\lim_{t\to\infty} \frac{U(t)}{a} \exp \frac{-\beta^2 t}{2} = e$$

The proof of the proposition is completed. From (6) it follows directly that

$$U(t) \leqq a \exp\left(\frac{\beta^2 t}{2} + 1\right) = ae \exp \frac{\beta^2 t}{2}$$

which implies (2) in virtue of (3).

The condition (8) shows that the estimation (2) with the constant *ae* is the best.

REFERENCE

[1] J. Hale, K. Meyer, *A class of functional equations of neutral type*, Memoirs of the A.M.S. 76 1967.