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Some Theorems on the Vector Product and Inner Product in an n-dimensional Euclidean Space

We meet in many books the following patterns:

$$a \wedge (b \wedge c) = \begin{vmatrix} b & c \\ a\,b & a\,c \end{vmatrix} \tag{1}$$

and

$$(a \wedge b)\cdot(c \wedge d) = \begin{vmatrix} a\cdot c & a\cdot d \\ b\cdot c & b\cdot d \end{vmatrix} \tag{2}$$

where a, b, c and d are arbitrary vectors in a 3-dimensional Euclidean space over R (the field of real numbers), $\wedge$ and $\cdot$ are symbols of the vector product and inner product respectively. In this paper we shall generalize these patterns in an n-dimensional Euclidean vector space V_n ($n \geqslant 3$) over R. We can find the fundamental properties of $\wedge$ and $\cdot$ in [1], [2], [3], [4] and others.

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I. INTRODUCTION

Let (e_i) $(i = 1, \dots, n)$ denotes an orthonormal basis of V_n, then every vector $a \in V_n$ can be written in the form

$$a = \alpha^i e_i \quad (\alpha^i \in R)\,. \tag{3}$$

Definition 1. For every two vectors a and b in V_n the expression $\sum_{i=1}^{n} \alpha^i \beta^i$ is called the scalar product of a and b and is denoted by $a\cdot b$, where $b = \beta^i e_i$.

Definition 2. a is perpendicular to b ($a \perp b$) if $a\cdot b = 0$.

Definition 3. The number $|a| = \sqrt{a \cdot a}$ is called the length of the vector a.
Definition 4. The set of all vectors of the form

$$\mu_1 a_1 + \mu_2 a_2 + \dots + \mu_m a_m \quad (0 \leqslant \mu_j \leqslant 1,\ j = 1, \dots, m,\ m \leqslant n)$$

is called the parallelepiped built on the vectors a_j and is denoted by $\pi(a_1, \dots, a_m)$; a_j are called edges of $\pi(a_1, \dots, a_m)$.

The length of vector a is usually called the 1-dimensional volume of $\pi(a)$. The area of the parallelogram built on a_1 and a_2 is called the 2-dimensional volume of $\pi(a_1, a_2)$ and is denoted by $V(a_1, a_2)$. To define the m-dimensional volume of $\pi(a_1, \dots, a_m)$ we call $\pi(a_1, \dots, a_{j-1}, a_{j-1}, \dots, a_m)$ $(1 \leqslant j \leqslant m)$ a base of the $\pi(a_1, \dots, a_m)$, and the component of a_j, which is perpendicular to all a_k $(k \neq j)$, the altitude of $\pi(a_1, \dots, a_m)$ corresponding to the base $\pi(a_1, \dots, a_{j-1}, a_{j-1}, \dots, a_m)$. This altitude is denoted by h_j. The recurrence formula

$$V(a_1) = |a_1|\,, \tag{4}$$

$$V(a_1, \dots, a_m) = V(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m) \cdot |h_j| \quad (2 \leqslant m \leqslant n) \tag{5}$$

is definite for $1 \leqslant m \leqslant n$, and $V(a_1, \dots, a_m)$ is called the m-dimensional volume of $\pi(a_1, \dots, a_m)$. Note it can be proved that $V(a_1, \dots, a_m)$ is not dependent on the choice of a_j (cf. chap. X, §3, [3]). If $a_k \perp a_j$ $(k \neq j)$, then $h_j = a_j$ and we have

$$V(a_1, \dots, a_m) = V(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_m) \cdot |a_j|\,. \tag{6}$$

Definition 5. A sequence of n linearly independent vectors a_i, $i = 1, \dots, n$, is oriented in accordance with the orientation defined by the basis (e_i) if $\det\|a_i^j\| > 0$, where $a_i = a_i^j e_j$.

Definition 6. Let (b_q) $(q = 1, \dots, n-1)$ be a sequence of $n-1$ vectors in V_n. The vector b of the following properties:

1. $|b| = V(b_1, \dots, b_{n-1})$,
2. $b \perp b_q (q = 1, \dots, n-1)$,
3. $(b_1, \dots, b_{n-1,b})$ is oriented in accordance with the orientation defined by the basis $(e_1, \dots, e_n)$,

is called the vector product of the sequence (b_q) and is denoted by $b_1 \wedge \dots \wedge b_{n-1}$.

We shall always use the symbol $b = b_1 \wedge \dots \wedge b_{n-1}$ in the next theorems. It is not difficult to show that the definite 6 is equivalent to others in [1], [4].

II. THEOREMS

We shall always allow p to run from 1 to $n-2$ and q from 1 to $n-1$ and denote $a = a_1 \wedge \dots \wedge a_{n-2} \wedge (b_1 \wedge \dots \wedge b_{n-1})$.

Theorem 1. Let (a_p) and (b_q) be sequences of vectors in V_n. Then

(7)
$$a = \begin{vmatrix} b_1 & b_2 & \dots & b_{n-1} \\ a_1\cdot b_1 & a_1\cdot b_2 & \dots & a_1\cdot b_{n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-2}\cdot b_1 & a_{n-2}\cdot b_2 & & a_{n-2}\cdot b_{n-1} \end{vmatrix} \cdot(-1)^{n-1}.$$

This theorem is a generalization of the pattern (1).

Proof. 1. In the case when b_q are linearly dependent we have $b = 0$ (cf. chap. III, § 44, 1°, [1]), so $a = 0$. Because b_q are linearly dependent, not restricting the generality, we can write

$$b_{n-1} = \sum_{r=1}^{n-2} \gamma_r b_r \quad \left(\sum |\gamma_r| \neq 0\right).$$

Then we have

$$a_p \cdot b_{n-1} = a_p \cdot \left(\sum \gamma_r b_r\right) = \sum \gamma_r (a_p \cdot b_r),$$

so the determinant on the right side of (7) is equal to 0. Therefore the right side of (7) is equal to the left one, q.e.d.

2. The case when b_q are linearly independent, i.e. $b \neq 0$, and b is linearly dependent on all a_p.

We have at once $a = 0$. We shall show that the right side of (7) is equal to 0. Obviously we can write

(8)
$$b = \sum_{p=1}^{n-2} \nu_p a_p \quad \left(\sum |\nu_p| \neq 0\right).$$

From definition 6 we have $b \perp b_q$ or

(9)
$$b \cdot b_q = \sum_{p=1}^{n-2} \nu_p (a_p \cdot b_q) = 0.$$

This system of $n-1$ equations and $n-2$ unknowns ν_p has a nonzero solution, hence, every determinant of the matrix

$$\left\| \begin{matrix} a_1\ b_1 & a_2\ b_1 & \dots & a_{n-2}\ b_1 \\ \dots & \dots & \dots & \dots \\ a_1\ b_{n-1} & a_2\ b_{n-1} & \dots & a_{n-2}\ b_{n-1} \end{matrix} \right\| \quad \text{(without the } q\text{-th row)}$$

is equal to 0 (cf. Theorem of Kronecker–Capelle [2]). And finally the right side of (7) is equal to 0, q.e.d.

3. The case when b_q are linearly independent, i.e. $b \neq 0$, and a_p, b are linearly independent, i.e. $a \neq 0$.

From definition 6 we have $a \perp b$ and $b \perp b_q$ for all q, hence a belongs to the $(n-1)$-dimensional subspace perpendicular to b and generated by all b_q and we can write

(10)
$$a = \sum_{q=1}^{n-1} \eta_q b_q \quad \left(\sum |\eta_q| \neq 0\right).$$

Using definition 6 we have

$$(11)\qquad a_p \cdot a = \sum_{q=1}^{n-1} \eta_q (a_p \cdot b_q) = 0$$

This system of $n-2$ homogeneous equations of $n-1$ unknowns η_q has a non-zero solution (from (10)), so among the determinants of the square matrix

$$(-1)^q \|a_p \cdot b_q\| \quad \text{(without the } q\text{-th column)}$$

which will be denoted by A_q, there is at least one different from 0. Not restricting the generality, let $A_{n-1} \neq 0$. Using Cramer's patterns and treating η_{n-1} as given, we get the solution

$$(12)\qquad \eta_p = \frac{\eta_{n-1} A_p}{A_{n-1}} = \lambda A_p\,, \quad \text{where } \lambda = \frac{\eta_{n-1}}{A_{n-1}}\,.$$

From (10) and (12) we have $a = \sum_{q=1}^{n-1} \lambda A_q b_q$ or writting in extenso

$$(13)\qquad a = \lambda \begin{vmatrix} b_1 & b_2 & \dots & b_{n-1} \\ a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_{n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-2} \cdot b_1 & a_{n-2} \cdot b_2 & \dots & a_{n-2} \cdot b_{n-1} \end{vmatrix}$$

We shall show that $\lambda = (-1)^{n-1}$.

Let us denote by $P_{n-1}(b)$ the $(n-1)$-dimensional subspace perpendicular to b. Not changing a, we can replace the sequence (a_p) by (a'_p) so that $a'_p \perp b$ and $a'_{p_1} \perp a'_{p_2}$ $(p_1 \neq p_2, p_1, p_2 = 1, \dots, n-2)$. Thus $a'_p \in P_{n-1}(b)$ and $a = a'_1 \wedge \dots \wedge a'_{n-2} \wedge b$.

Not restricting the generality and not changing a and b, we can replace the sequence (b_q) by (b'_q) so that $b'_p = \tau_p a'_p$ $(\tau_p > 0)$ and $b'_p \perp b'_{n-1}$. Thus $a = a'_1 \wedge \dots \wedge a'_{n-2} \wedge (b'_1 \wedge \dots \wedge b'_{n-1})$. Therefore and from (13) and because λ is a scalar, we obtain

$$a = \lambda \begin{vmatrix} b'_1 & b'_2 & \dots & b'_{n-1} \\ a'_1\, b'_1 & a'_1\, b'_2 & \dots & a'_1\, b'_{n-1} \\ \dots & \dots & \dots & \dots \\ a'_{n-2}\, b'_1 & a'_{n-2}\, b'_2 & \dots & a'_{n-2}\, b'_{n-1} \end{vmatrix}$$

$$= \lambda \tau_1 \dots \tau_{n-2} (-1)^n \begin{vmatrix} a'_1\, a'_1 & 0 & \dots & 0 \\ 0 & a'_2 a'_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a'_{n-2}\, a'_{n-2} \end{vmatrix} b'_{n-1}$$

So we get

$$(14)\qquad a = \lambda \tau_1 \dots \tau_{n-2} (-1)^n V^2(a'_1, \dots, a'_{n-2}) b'_{n-1}\,.$$

(cf. Theorem 7, chap. X, § 3, 3).

On the other hand, from the definition of a'_p and b'_p, we see that the vectors b'_{n-1} and a both belong to $P_{n-1}(b)$, $b'_{n-1} \perp a'_p$, $b'_{n-1} \perp b$ and $a \perp a'_p$, $a \perp b$, hence, they must both belong to a $(n-1)$-dimensional subspace contained in $P_{n-1}(b)$. But they have different orientations, unless there is a contradiction to 3. of definition 6, so we have $a = -\sigma b'_{n-1}$ $(\sigma > 0)$ or $|a| = \sigma |b'_{n-1}|$. From definition 6 and (6) we get also

$$\begin{aligned} |a| &= V(a'_1, \dots, a'_{n-2}, b) \\ &= V(a'_1, \dots, a'_{n-2})|b| \\ &= V(a'_1, \dots, a'_{n-2})\, V(b'_1, \dots, b'_{n-1}) \\ &= V(a'_1, \dots, a'_{n-2})\, V(b'_1, \dots, b'_{n-2})|b'_{n-1}| \\ &= \tau_1 \dots \tau_{n-2} V^2(a'_1, \dots, a'_{n-2})|b'_{n-1}|, \end{aligned}$$

(cf. Theorem 9, chap. X, § 3, [3]), hence $\sigma = \tau_1 \dots \tau_{n-2} V^2(a'_1, \dots, a'_{n-2})$ or

$$(15) \qquad a = -\tau_1, \dots, \tau_{n-2} V^2(a'_1, \dots, a'_{n-2})\; b'_{n-1}.$$

From (14) and (15) is obtained the result $\lambda = (-1)^{n-1}$, q.e.d.

Theorem 2. Let (a_q) and (b_q) be two sequences of $n-1$ vectors in V_n. Then

$$(16) \qquad (a_1 \wedge \dots \wedge a_{n-1}) \cdot (b_1 \wedge \dots \wedge b_{n-1}) = \det \|a_k \cdot b_l\|, \ (k, l = 1, \dots, n-1).$$

Proof. Let (e_i) be a canonical (orthonormal) basis of the space V_n. Using the definitions 5 and 6, we get without difficulty

$$(17) \qquad e_t = (-1)^{n-t} e_1 \wedge \dots \wedge e_{t-1} \wedge e_{t-1} \wedge \dots \wedge e_n \quad (t = 1, \dots, n).$$

Suppose that we are given a sequence of $n-1$ indices i_k $(i_k = 1, \dots, n)$ ordered in such a manner that $i_1 < \dots < i_{n-1}$, then there exists exactly one number $t \in \{1, \dots, n\}$ such that

$$(18) \qquad e_{i_1} \wedge \dots \wedge e_{i_{n-1}} = e_1 \wedge \dots \wedge e_{t-1} \wedge e_{t+1} \wedge \dots \wedge e_n = (-1)^{n-t} e_t.$$

Now we can write, from (3)

$$\begin{aligned} (19) \quad & (a_1 \wedge \dots \wedge a_{n-1}) \cdot (b_1 \wedge \dots \wedge b_{n-1}) \\ &= \big((\alpha_1^{i_1} e_{i_1}) \wedge \dots \wedge (\alpha_{n-1}^{i_{n-1}} e_{i_{n-1}})\big) \cdot \big((\beta_1^{j_1} e_{j_1}) \wedge \dots \wedge (\beta_{n-1}^{j_{n-1}} e_{j_{n-1}})\big) \\ &= \alpha_1^{[i_1} \dots \alpha_{n-1}^{i_{n-1}]} \beta_1^{[j_1} \dots \beta_{n-1}^{j_{n-1}]} (e_{i_1} \wedge \dots \wedge e_{i_{n-1}}) \cdot (e_{j_1} \wedge \dots \wedge e_{j_{n-1}}), \end{aligned}$$

where $i_1 < \dots < i_{n-1}$, $j_1 < \dots < j_{n-1}$.

According to (18) this can be put as

$$e_{i_1} \wedge \dots \wedge e_{i_{n-1}} = (-1)^{n-t} e_t \ \text{and} \ e_{j_1} \wedge \dots \wedge e_{j_{n-1}} = (-1)^{n-s} e_s,$$

hence, denoting $\alpha^t = \alpha_1^{[i_1} \dots \alpha_{n-1}^{i_{n-1}]}$ and $\beta^s = \beta_1^{[j_1} \dots \beta_{n-1}^{j_{n-1}]}$, we get

$$\begin{aligned} (20) \quad (a_1 \wedge \dots \wedge a_{n-1}) \cdot (b_1 \wedge \dots \wedge b_{n-1}) &= \sum_{t,s1}^{n} (-1)^{t+s} \alpha^t \beta^s e_t \cdot e_s \\ &= \sum_{t,s1}^{n} \alpha^t \beta^t \ (\text{because } e_t \cdot e_s = \delta_s^t). \end{aligned}$$

On the other hand, since $e_i \cdot e_j = \delta^i_j$, the matrix $\|e_i \cdot e_j\|$ is the unit matrix. This allows us to write

$$\det\|e_{i_k} \cdot e_{j_l}\| = \varepsilon^{i_1 \dots i_{n-1}}_{j_1 \dots j_{n-1}} = \det\|\delta^{i_k}_{j_l}\| \quad (i_k, j_l = 1, \dots, n)\,. \tag{21}$$

Finally we have as an immediate consequence of (21) and the properties of determinants

$$\det\|a_k \cdot b_l\| = \det\|(\overset{i_k}{k} e_{i_k}) \cdot (\overset{j_l}{l} e_{j_l})\| = \sum_{\substack{i_1 < \dots < i_{n-1} \\ j_1 < \dots < j_{n-1}}} \alpha_1^{[i_1} \dots \alpha_{n-1}^{i_{n-1}]} \beta_1^{[j_1} \dots \beta_{n-1}^{j_{n-1}]} \det\|e_{i_k} \cdot e_{j_l}\|$$

$$= \sum_{i_1 < \dots < i_{n-1}} \alpha_1^{[i_1} \dots \alpha_{n-1}^{i_{n-1}]} \beta_1^{[i_1} \dots \beta_{n-1}^{i_{n-1}]} = \sum_{t=1}^{n} \alpha^t \beta^t$$

$$= (a_1 \wedge \dots \wedge a_{n-1}) \cdot (b_1 \wedge \dots \wedge b_{n-1}) \text{ (from (20)), q.e.d.}$$

III. FINAL REMARKS

Corollary 1. If we replace the q-th vector of the product $a_1 \wedge \dots \wedge a_q \wedge \dots \wedge a_{n-1}$ by an other vector product $b_1 \wedge \dots \wedge b_{n-1}$, then we get

$$(a_1 \wedge \dots \wedge a_{q-1} \wedge (b_1 \wedge \dots \wedge b_{n-1}) \wedge a_{q+1} \wedge \dots \wedge a_{n-1} = (-1)^q \begin{vmatrix} b_1 & b_2 & \dots & b_{n-1} \\ a_1 \cdot b_1 & a_1 \cdot b_2 & \dots & a_1 \cdot b_{n-1} \\ \dots & \dots & \dots & \dots \\ a_{q-1} \cdot b_1 & a_{q-1} \cdot b_2 & \dots & a_{q-1} \cdot b_{n-1} \\ a_{q+1} \cdot b_1 & a_{q+1} \cdot b_2 & \dots & a_{q+1} \cdot b_{n-1} \\ \dots & \dots & \dots & \dots \\ a_{n-1} \cdot b_1 & a_{n-1} \cdot b_2 & \dots & a_{n-1} \cdot b_{n-1} \end{vmatrix}. \tag{22}$$

Proof. Using the antisymmetry of the vector product and Theorem 1 we get at once the result (22).

Corollary 2. If we replace every vector a_q of the vector product $a_1 \wedge \dots \wedge a_{n-1}$ by a vector product $\overset{q}{c}_1 \wedge \dots \wedge \overset{q}{c}_{n-1}$ and if we fix a number k from $1, \dots, n-1$, then we get

$$(\overset{1}{c}_1 \wedge \dots \wedge \overset{1}{c}_{n-1}) \wedge \dots \wedge (\overset{n-1}{c}_1 \wedge \dots \wedge \overset{n-1}{c}_{n-1}) = (-1)^k \begin{vmatrix} \overset{k}{c}_1 & \overset{k}{c}_2 & \dots & \overset{k}{c}_{n-1} \\ V_{11} & V_{12} & & V_{1\,n-1} \\ \dots & \dots & \dots & \dots \\ V_{k-1\,1} & V_{k-1\,2} & \dots & V_{k-1\,n-1} \\ V_{k+1\,1} & V_{k+1\,2} & \dots & V_{k+1\,n-1} \\ \dots & \dots & \dots & \dots \\ V_{n-1\,1} & V_{n-1\,2} & \dots & V_{n-1\,n-1} \end{vmatrix}, \text{ for all } k = 1, \dots, n-1\,, \tag{23}$$