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Erwin Turdza

On a Functional Inequality with the *n*-th Iterate of the Unknown Function

In this paper we shall deal with solutions of the inequality

$$\Psi^n(x) \geqslant g(x) \tag{1}$$

in the class $C(a, b)$ of the functions Ψ: $(a, b) \to (a, b)$, continuous in the interval (a, b). Ψ^n denotes here n-th iterate of the function Ψ. In the present paper we give some conditions sufficient for a function $\Psi \in C(a, b)$ to be a solution of the inequality (1) for a positive integer n.

In the sequel we shall assume the following condition (H). The function g is strictly increasing and continuous in the interval $[a, b]$, $g(a) = a$, $g(b) = b$, $g(x) > x$ for $x \in (a, b)$.

Under the hypothesis (H) the equation

$$\varphi^n(x) = g(x) \tag{2}$$

where φ^n denotes the n-th iterate of the function φ, has in the interval (a, b) a continuous solution $\varphi \in C(a, b)$ depending on an arbitrary function and all the continuous solutions of (2) are strictly increasing in the interval (a, b) (see [1] cf. also [2], p. 297).

Lemma 1. If there exists a positive integer n such that $\Psi \in C(a, b)$ is a continuous solution of the inequality (1) and the hypothesis (H) is fulfilled, then

$$\Psi(x) > x \quad \text{for } x \in (a, b)\,. \tag{3}$$

Proof. If the lemma is false, then either $\Psi(x) < x$ for $x \in (a, b)$ or there exists an $x_0 \in (a, b)$ such that $\Psi(x_0) = x_0$. In the first case we have $\Psi^k(x) < x$ for $x \in (a, b)$, $k = 1, 2, \ldots$ and in particular $\Psi^n(x) < x < g(x)$ for $x \in (a, b)$, which is impossible. In the second case we have $\Psi^k(x_0) = x_0$ for $k = 1, 2, \ldots$ and in particular $\Psi^n(x_0) = x_0 < g(x_0)$ which is also impossible.

Corollary 1. If the continuous function $\Psi \in C(a, b)$ fulfils the inequality (1) in the interval (a, b) for a positive integer n, then

$$\Psi^k(x) \geqslant g(x) \quad \text{for } x \in (a, b),\ k \geqslant n\,.$$

Corollary 2. If the continuous function $\Psi \in C(a, b)$ fulfils the inequality (1) in the interval (a, b) for a positive integer n, then

$$\lim_{x\to b} \Psi(x) = b .$$

Proof. Let the continuous function $\Psi \in C(a, b)$ be a solution of the inequality (1) for some positive integer n. Hence and from lemma 1 we have $x < \Psi(x) < b$ for $x \in (a, b)$, which implies that $\lim_{x\to b} \Psi(x) = b$.

Lemma 2. Let the function $\Psi \in C(a, b)$ fulfil the inequality (1) in the interval (a, b) for a certain positive integer n and let a function $\Phi \in C(a, b)$ fulfil the inequality

$$\Phi(x) \geqslant \Psi(x) \quad \text{for } x \in (a, b) . \tag{4}$$

If either the function Ψ or Φ is increasing in the interval (a, b), then

$$\Phi^n(x) \geqslant g(x) \quad \text{for } x \in (a, b) .$$

Proof. Let the function Ψ be increasing in the interval (a, b). The inequality (4) implies

$$\Phi^n(x) \geqslant \Psi\big(\Phi^{n-1}(x)\big) \geqslant \Psi^2\big(\Phi^{n-1}(x)\big) \geqslant \ldots \geqslant \Psi^n(x) \geqslant g(x) \quad \text{for } x \in (a, b) .$$

If the function Φ is increasing in the interval (a, b), then we have from (4), $\Phi^2(x) \geqslant \Phi\big(\Psi(x)\big)$ for $x \in (a, b)$. Hence, by virtue of (4), we have $\Phi^2(x) \geqslant \Psi^2(x)$ for $x \in (a, b)$. Repeating this reasoning $n-2$ times we obtain $\Phi^n(x) \geqslant \Psi^n(x) \geqslant g(x)$ for $x \in (a, b)$.

Corollary 3. If the continuous function $\varphi \in C(a, b)$ fulfils the equation (2) for a certain positive integer n and the continuous function $\Psi \in C(a, b)$ fulfils the inequality

$$\Psi(x) \geqslant \varphi(x) \quad \text{for } x \in (a, b) ,$$

then the function Ψ fulfils the inequality (1) for $x \in (a, b)$.

In a manner similar to lemma 2, we can prove the following

Lemma 3. Let the functions Φ and Ψ be continuous in the interval (a, b) and let the inequality $\Phi(x) \leqslant \Psi(x)$ be fulfilled for $x \in (a, b)$. If either the function Ψ or Φ is increasing in the interval (a, b), then

$$\Phi^k(x) \leqslant \Psi^k(x) \quad \text{for } x \in (a, b) , \quad k = 1, 2, \ldots$$

In particular, if for some positive integer n and some $x_0 \in (a, b)$ we have $\Psi^n(x_0) < g(x_0)$, then $\Phi^n(x_0) < g(x_0)$.

Theorem 1. Let the function g fulfil the hypothesis (H) and let for a function $\Psi \in C(a, b)$ the inequality (3) be fulfilled. If there exist real numbers $a < a' < b' < b$ and a positive integer p such that

$$\Psi^p(x) \geqslant g(x) \quad \text{for } x \in (a, a') \cup (b', b) , \tag{5}$$

then there exists a positive integer $n \geqslant p$ such that the inequality (1) holds for $x \in (a, b)$.

Proof. From (3) we have $\Psi^l(x) \geqslant \Psi^{l-1}(x) \geqslant \dots \geqslant \Psi(x)$ for $x \in (a, b)$, $l = 1, 2, \dots$, whence

$$\Psi^l(x) \geqslant \Psi^p(x) \geqslant g(x) \quad \text{for } x \in (a, a') \cup (b', b),\ l \geqslant p\,. \tag{6}$$

Let us put $k(x) = \Psi(x) - x$, $\mathbf{k} = \inf\limits_{x \in [a', b']} (x)$. It follows from (3) that $k(x) > 0$ for $x \in (a, b)$, and then $\mathbf{k} > 0$. Let $x \in [a', b']$ and let s_x be a positive integer such that $x + s_x\, \mathbf{k} \in (b', b)$. From that and from (3) we have

$$\Psi^{s_x}(x) = x + k(x) + k\big(x + k(x)\big) + \dots + k\big(x + k(x) + \dots + k(x + \dots)\big) = x + s_x \mathbf{k} + r(x)\,.$$

Since $k(x) > 0$ and $\Psi\big((a, b)\big) \subset (a, b)$, then $r(x) \geqslant 0$ and $x + s_x\mathbf{k} + r(x) \in (b', b)$. This together with (3), (5) and the monotonicity of the function g yields

$$\Psi^{s_x+1+p}(x) > \Psi^p\big(x + s_x\mathbf{k} + r(x)\big) \geqslant g\big(x + s_x\mathbf{k} + r(x)\big) \geqslant g(x) \quad \text{for } x \in [a', b']\,. \tag{7}$$

Let t be a positive integer such that $t\mathbf{k} \geqslant b' - a'$. Then from (3) and (7) we have

$$\Psi^{t+1+p}(x) \geqslant \Psi^{s_x+1+p}(x) \geqslant g(x) \quad \text{for } x \in [a', b']\,. \tag{8}$$

It follows from (6) and (8) that the inequality (1) holds for $n = t + 1 + p$, $x \in (a, b)$. This complete the proof.

Remark. The following example proves that the assumption (5) of the theorem 1 cannot be omitted. Let us put $(a, b) = (0, 1)$, $g(x) = x^{1/2}$, $\Psi(x) = \frac{1}{2}x^2 + \frac{1}{2}$. Since for each positive integer k we have $(\Psi^k)'(1) = 1$ and $g'(1) = \frac{1}{2}$, then there exists a left-hand neighbourhood U_k of the point 1 such that $\Psi^k(x) < g(x)$ for $x \in U_k$. Then the inequality (1) does not hold in $(0, 1)$ for any positive integer k.

Theorem 2. Let the function g fulfil the hypothesis (H), let the function $\Psi \in C(a, b)$ fulfil the inequality (3) for $x \in (a, b)$ and let $\lim\limits_{x \to a+} \Psi(x) = a$. If there exist real numbers $m_1, m_2, \mu_1, \mu_2, a', b'$, such that $\mu_1, m_1 > 1$, $\mu_2, m_2 \in (0, 1)$, $a < a' < b' < b$ and

$$g(x) \leqslant m_2 x + b(1 - m_2) \quad \text{for } x \in (b', b)\,, \tag{9}$$

$$\Psi(x) \geqslant \mu_2 x + b(1 - \mu_2) \quad \text{for } x \in (b', b)\,, \tag{10}$$

$$g(x) \leqslant m_1 x + a(1 - m_1) \quad \text{for } x \in (a, a')\,, \tag{11}$$

$$\Psi(x) \geqslant \mu_1 x + a(1 - \mu_1) \quad \text{for } x \in (a, a')\,, \tag{12}$$

then there exists a positive integer n such that the inequality (1) holds for $x \in (a, b)$.

Proof. It follows from (3) that $\Psi\big((b', b)\big) \subset (b', b)$, whence, for $x \in (b', b)$, we have

$$\begin{aligned} \Psi^k(x) &\geqslant \mu_1 \Psi^{k-1}(x) + b(1 - \mu_2) \geqslant \mu_2\big(\mu_2 \Psi^{k-2}(x) + b(1 - \mu_2)\big) + b(1 - \mu_2) \geqslant \dots \\ &\geqslant \mu_2^k x + \mu_2^{k-1} b(1 - \mu_2) + \dots + b(1 - \mu_2) = \mu_2^k x + b(1 - \mu_2^k)\,. \end{aligned}$$

Since $\mu_2, m_2 \in (0, 1)$, there exists a positive integer k_1 such that $\mu_2^{k_1} \leqslant m_2$, whence $\mu_2^k x+b(1-\mu_2^k) \geqslant m_2 x+b(1-m_2) \geqslant g(x)$ for $x(b', b)$, $k \geqslant k_1$. Since $\mu_1, m_1 > 1$, then there exists a positive integer k_2 such that $\mu_1^{k_2} \geqslant m_1$. Let us denote $p = \max(k_1, k_2)$. Since $\Psi \in C(a, b)$ and $\lim\limits_{x\to a+} \Psi(x) = a$, then there exists an a'' $\in (a, a')$ such that $\Psi^p\big((a, a'')\big) \subset (a, a')$, whence, for $x(a, a'')$, we obtain

$$\Psi^p(x) \geqslant \mu_1 \Psi^{p-1}(x)+b(1-\mu_1) \geqslant \ldots \geqslant \mu_1^p x+\mu_1^{p-1} b(1-\mu_1)+\ldots+b(1-\mu_1) =$$
$$= \mu_1^p x+b(1-\mu_1^p) \geqslant m_1 x+b(1-m_1) \geqslant g(x) \quad \text{for } x \in (a, a'') .$$

Let us denote $\overline{\Psi}(x) = \Psi^p(x)$ for $x \in (a, b)$. The function $\overline{\Psi}$ fulfils the assumptions of theorem 1, then there exists a positive integer r such that $\overline{\Psi}^r(x) \geqslant g(x)$ for $x \in (a, b)$. Whence, putting $n = pr$ we obtain $\Psi^n(x) \geqslant g(x)$ for $x \in (a, b)$, and the proof is concluded.

Remark. Since $x^{1/2} \leqslant \frac{1}{2}x^2+\frac{1}{2}$ for $x \in (0, 1)$ and for each $\mu \in (0, 1)$ there exists a left-hand neighbourhood U_μ of the point 1 such that $\frac{1}{2}x^2+\frac{1}{2} < \mu x+1-\mu$ for $x \in U_\mu$, then the example 1 proves that the assumption (10) of the theorem 2 cannot be omitted. It is easy to construct analogous examples showing that the assumptions (9), (10), and (12) of the theorem 2 cannot be omitted.

Lemma 4. Let a function $\chi \in C[a, b]$ satisfy the conditions

$$\chi(x) > x \text{ for } x \in (a, b),\ \chi(a) = a,\ \chi(b) = b$$

and let $h_1, h_2 \in C[a, b]$ be two increasing functions in $[a, b]$. If there are numbers a', b', $b > a' > a$ and $a < b' < b$ such that

$$\chi(x) \geqslant h_1(x) \quad \text{for } x \in [a, a') \tag{13a}$$

$$\chi(x) \geqslant h_2(x) \quad \text{for } x \in (b', b] \tag{13b}$$

then there exist an $a_1 \in (a, a')$ and a function $\Psi \in C[a, b]$ which is increasing in $[a, b]$ and fulfils the conditions:

$$\Psi(x) \leqslant \chi(x) \quad \text{for } x \in [a, b] \tag{14}$$

and

$$\Psi(x) \geqslant h_1(x) \quad \text{for } x \in [a, a_1) \tag{15a}$$

$$\Psi(x) \geqslant h_2(x) \quad \text{for } x(b', b] . \tag{15b}$$

Proof. We shall prove that the function

$$\Psi(x) = \inf_{t \in [x, b)} \chi(t) \tag{16}$$

has all the required properties. In fact, it is increasing in $[a, b)$ and fulfils (14) (see [2], p. 71).

Moreover, from the monotonicity of the function h_2 and (13b) we have $\chi(t) \geqslant h_2(t)$ for $t \in [x, b)$ and $x \in (b', b]$. Taking infimum we get (15b) since $\inf\limits_{t \in [x, b)} h_2(t) = h_2(t)$.

To prove (15a) first denote $u = \inf_{t\in[a',b]} \chi(t)$ and put

$$a_1 = \sup\{x \in (a, a'] \colon \inf_{t\in[x,a)} \chi(t) \leqslant u\}.$$

Since $\chi(x) > x$, then $u > a' > a$ and from the continuity of the function χ we infer that a_1 actually exists.

Now, for every $x \in [a, a_1)$ we have

$$\inf_{t\in[x,a_1)} \chi(t) = \inf_{t\in[x,a')} \chi(t) \leqslant \inf_{t\in[x,b]} \chi(t) = \Psi(x) . \tag{17}$$

By (16), and from (13a) we see that

$$\inf_{t\in[x,a')} \chi(t) \geqslant \inf_{t\in[x,a')} h_1(t) = h_1(x) , \tag{18}$$

as h_1 is increasing in $[a, a')$.

Inequalities (17) and (18) yield (15a) and the proof of the lemma is completed.

Theorem 3. Let the function g fulfil the hypothesis (H). Let a function Ψ satisfy the conditions

$$\Psi \in C[a, b],\ \Psi(x) > x \quad \text{for } x \in (a, b),\ \Psi(a) = a,\ \Psi(b) = b ,$$

and let there exist real numbers $\alpha, \beta \in (0, 1)$ and a', b' such that

$$\Psi(x) > \alpha g(x) + (1-\alpha)x \quad \text{for } x \in (a, a') , \tag{19}$$

$$\Psi(x) > \beta g(x) + (1-\beta)x \quad \text{for } x \in (b', b) . \tag{20}$$

If there exists a positive number s such that

$$g(y) - g(x) \geqslant s(y-x) \quad \text{for } y > x \text{ and } x, y \in (a, a') \text{ or } x, y \in (b', b) , \tag{21}$$

then there exists a positive integer n such that the inequality (1) holds for $x \in (a, b)$.

Proof. The lemmas 2 and 4 imply that it is enough to prove the theorem for the function Ψ increasing. Let us denote

$$\eta = 1-\beta$$

and

$$\varepsilon(x) = g\big(\beta g(x) + \eta x\big) - g(x) .$$

We shall prove by induction the following inequality

$$\Psi^n(x) > (1-\eta^n) g(x) + \eta^n x + (1-\eta^{n-1})\varepsilon(x) + \ldots + (1-\eta)\varepsilon\big(\Psi^{n-2}(x)\big) \quad \text{for } x \in (b', b) \tag{22}$$

Since $\Psi^l((b', b)) \subset (b', b)$ for $l = 1, 2, \ldots$, then from (20) and from the condition $\Psi(x) > x$ for $x \in (a, b)$ and from the monotonicity of the functions g and Ψ we have

$$\Psi^2(x) > \beta g\big(\Psi(x)\big) + \eta \Psi(x) > \beta g\big(\beta g(x) + \eta x\big) + \eta\big(\beta g(x) + \eta x\big)$$
$$= (\beta + \eta\beta) g(x) + \eta^2 x + \beta\varepsilon(x) = (1-\eta^2) g(x) + \eta^2 x + (1-\eta)\varepsilon(x) \quad \text{for } x \in (b', b) .$$