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Special Solutions of a Linear Functional Equation

§ 1. INTRODUCTION

In the present note we are concerned with the linear functional equation

$$(L) \quad \varphi(f(x)) = g(x)\varphi(x) + h(x),$$

where φ is the unknown function, x is a real variable, and all functions appearing in (L) are real and defined in an interval $I = \langle 0, a \rangle$, $a > 0$. This note is connected with paper [7] dealing with the problem of the existence of local analytic solutions of equation (L) if the given functions have zeros of some orders at $\xi = 0$, ξ being the fixed point of f .

We shall be interested in real solutions of equation (L) in a class of power-type functions in a neighbourhood of the origin, i.e. a class related to that U^a introduced by M. Kuczma [5]. Our considerations are based on the theory of continuous solutions of equation (L) contained mainly in [6], [2].

§ 2. PRELIMINARIES

We start with the definition of the class V^a .

Definition 1. For every a positive we denote by V^a the class of functions φ which are defined and continuous in a neighbourhood of zero of the form $(0, \delta)$ and for which there exists a $\lim_{x \rightarrow 0} x^{-a}\varphi(x) \neq 0$.

Looking for solutions in the class V^a it is natural to assume that the functions f, g, h belong to some classes V^a (in particular $f \in V^p$, $p > 1$, $g \in V^q$, $q > 0$). This

corresponds to the case $f'(0) = 0$, $g(0) = 0$ (other cases have been studied in [5]).

We shall make the following assumptions.

- (i) The function $f(x)$ is continuous and strictly increasing in $I = \langle 0, a \rangle$, $0 < f(x) \neq x$ for $x \neq 0$, and there exists a $p > 1$ such that $f \in V^p$.
- (ii) The function $g(x)$ is continuous in I and there exists a $q > 0$ such that $g \in V^q$.
- (iii) The function $h(x)$ is continuous in I and there exists a $r > 0$ such that $h \in V^r$.

Let us first note that from hypotheses (i)–(iii) it follows in particular that we may write

$$(1) \quad f(x) = x^p F(x), \quad g(x) = x^q G(x), \quad h(x) = x^r H(x),$$

$F(x)$, $G(x)$, $H(x)$ being continuous functions in I
and

$$(2) \quad F(0) \neq 0, \quad G(0) \neq 0, \quad H(0) \neq 0.$$

According to definition 1

$$(3) \quad \varphi \in V^a \Leftrightarrow \exists_{\delta > 0} \varphi: (0, \delta) \rightarrow \mathbb{R}, \quad \exists \Phi(x) = x^a \Phi(x), \quad \Phi(0) \neq 0, \quad \Phi \in C \langle 0, \delta \rangle$$

and we get inserting (1) into (L)

$$(4) \quad x^{pa} (F(x))^a \Phi(f(x)) = x^{a+q} G(x) \Phi(x) + x^r H(x).$$

It is readily seen that equation (L) has a solution φ in the class V^a if and only if equation (4) has a solution Φ continuous in a neighbourhood of zero. To designate this situation the following definition will be useful in the sequel.

Definition 2. We say that a functional equation (F_1) is locally equivalent to a functional equation (F_2) when a function $\varphi(x)$ is in a neighbourhood of zero a solution of equation (F_1) in the class V^a if and only if the function $\Phi(x) = x^{-a} \varphi(x)$ is in the neighbourhood a solution of equation (F_2) continuous and different from zero for $x = 0$.

Thus the investigation of the solutions in the class V^a of equation (L) can be reduced to studying continuous solutions of the auxiliary equation (4). In the theory of continuous solutions of equation (L) one uses the expression a "solution depending on an arbitrary function" (cf. the definition in [6], p. 45).

Definition 3. We say that equation (L) has in an interval I a continuous solution depending on an arbitrary function if there exists an interval $J \subset I$ such that every continuous function on J can be extended to a continuous solution of equation (L) in I .

We assume finally that a neighbourhood of zero is meant as an interval of the form $\langle 0, \delta \rangle$. Consequently, " $x \rightarrow 0$ " means " $x \rightarrow 0+$ ", and the 0-symbol refers to $x \rightarrow 0+$.

§ 2. RESULTS

Comparing the orders of the zeros of (4) we see that one of the following four cases must occur, provided that equation (L) has a solution $\varphi \in V^a$:

$$pa = r < q + a,$$

$$pa > q + a = r,$$

$$pa = q + a < r,$$

$$(pa = q + a = r,$$

We shall consider separately the resulting four conditions.

$$(C_1) \quad pq > r(p-1), \quad \text{and} \quad a = \frac{r}{p},$$

$$(C_2) \quad pq < r(p-1), \quad \text{and} \quad a = r - q,$$

$$(C_3) \quad pq < r(p-1), \quad \text{and} \quad a = \frac{q}{p-1},$$

$$(C_4) \quad pq = r(p-1), \quad \text{and} \quad a = \frac{q}{p-1}.$$

In case (C₁) we have the following

Theorem 1. If hypotheses (i)–(iii) are fulfilled and condition (C₁) holds, then equation (L) has in a neighbourhood $\langle 0, \delta \rangle$ of zero a solution in V^a , $a = \frac{r}{p}$, depending on an arbitrary function (cf. def. 3).

Proof. Under hypotheses (i)–(iii) equation (L) is locally equivalent (definition 2) to equation (4) which in view of (C₁) leads to

$$F(x)^a \Phi(f(x)) = x^{q+a-r} G(x) \Phi(x) + H(x),$$

i.e.

$$(5) \quad \Phi(f(x)) = \frac{x^{q+a-r} G(x)}{F(x)^a} \Phi(x) + \frac{H(x)}{F(x)^a}.$$

Let us note that in (5) the function $\gamma(x) \stackrel{\text{def}}{=} x^{q+a-r} G(x) F(x)^{-a}$ is by (C₁) continuous and different from zero in a neighbourhood of zero of the form $(0, \delta)$ and $\gamma(0) = 0$. Let $\langle 0, \delta \rangle$ be the interval in which the function $H(x) F(x)^{-a}$ is also continuous. In $\langle 0, \delta \rangle$ there are fulfilled the hypotheses of theorem 2.10 in [6] (cf. also [4]), whence it follows by $\gamma(0) = 0$ that equation (5) has in $\langle 0, \delta \rangle$ a continuous solution depending on an arbitrary function. Thus on account of the local equivalence of (5) and (L) equation (L) has in $\langle 0, \delta \rangle$ a solution in V^a depending on an arbitrary function. This completes the proof.

In case (C₂) we shall prove the following

Theorem 2. If hypotheses (i)–(iii) are fulfilled and condition (C_2) holds, then equation (L) has in a neighbourhood of zero a unique solution in the class V^a , $a = r - q$.

Proof. Under hypotheses (i)–(iii) equation (L) is locally equivalent to equation (4) which in view of (C_2) leads to

$$x^{pa-r} F(x)^a \Phi(f(x)) = G(x) \Phi(x) + H(x),$$

i.e.

$$\Phi(x) = \frac{x^{pa-r} F(x)^a}{G(x)} \Phi(f(x)) - \frac{H(x)}{G(x)}.$$

Denoting

$$\gamma(x) \stackrel{\text{df}}{=} \frac{x^{pa-r} F(x)^a}{G(x)}, \quad \chi(x) \stackrel{\text{df}}{=} -\frac{H(x)}{G(x)},$$

we rewrite the last equation in the form

$$(6) \quad \Phi(x) = \gamma(x) \Phi(f(x)) + \chi(x).$$

The function $\gamma(x)$ is continuous in a neighbourhood of zero and $\gamma(0) = 0$. Also $\chi(x)$ is continuous in a neighbourhood of zero. Let $\langle 0, \delta \rangle$ be the interval in which both the functions $\gamma(x)$ and $\chi(x)$ are continuous.

For equation (6) considered in $\langle 0, \delta \rangle$ in view of $\gamma(0) = 0$ there are fulfilled in fact the hypotheses of theorem 2.7 in [6] (cf. also [4]). So equation (6) has a unique continuous solution in $\langle 0, \delta \rangle$, and the assertion follows from the local equivalence of equations (6) and (L).

Before we come to the next cases we shall accept something more than relations $f \in V^p$ and $g \in V^q$ in (i)–(ii) assuming the following hypothesis.

(iv) There exist numbers $c > 0$, $d \neq 0$, $\mu > 0$, $\nu > 0$ and a bounded function $u(x)$ such that

$$(7) \quad f(x) = cx^p - x^{p+\mu} u(x), \quad x \in I,$$

and

$$(8) \quad g(x) = dx^q + o(x^{q+\nu}), \quad x \rightarrow 0.$$

The lemma below results from [8], theorem 2.2.

Lemma. If hypotheses (i) and (iv) for f are fulfilled, then for every number $b < a$ there exist a constant $0 < k < 1$ and $N(b)$ such that we have the estimation

$$(9) \quad f^n(x) \leq k^{p^{n+1}} \quad \text{for } x \in \langle 0, b \rangle \subset I, \quad n \geq N(b),$$

$f^n(x)$ being the n -th iterate of the function $f(x)$.

Now we are able to pass to the next case.

In case (C_2) equation (4) leads to

$$(10) \quad \Phi(f(x)) = \frac{G(x)}{F(x)^a} \Phi(x) + \frac{x^{-pa} H(x)}{F(x)^a}.$$

Let us first note that by (C_3) the equality $G(0) = F(0)^a$, in view of the local equivalence of (10) and (L), is the necessary condition for equation (L) to have a solution in the class V^a .

Theorem 3. If hypotheses (i)–(iv) are fulfilled, $d = c^a$, $a = \frac{q}{p-1}$, and condition (C_3) holds, then equation (L) has in the neighbourhood of zero a one-parameter family of solutions in the class V^a . These solutions are given by the formula

$$\varphi(x) = x^\alpha \Phi_\beta(x) = x^{\frac{q}{p-1}} \Phi_\beta(x), \quad \beta \neq 0,$$

where, for every $\beta \in R$, $\Phi_\beta(x)$ is in the neighbourhood the continuous solution of equation (10) such that $\Phi_\beta(0) = \beta$.

Proof. Equation (L) is now locally equivalent to equation (10). We shall prove that equation (10) has in a neighbourhood of zero a one-parameter family of continuous solutions. To prove it we shall show that the given functions of equation (10) fulfil the hypotheses of theorem 6 in [2].

It follows from (iv) with $d = c^a$, $a = \frac{q}{p-1}$, that

$$\frac{G(0)}{F(0)^a} = 1.$$

For, by (1) together with (7) and by (1) together with (8) we have

$$(11) \quad F(x) = c - x^\mu u(x), \quad x \in I,$$

and

$$(12) \quad G(x) = d + 0(x^\nu), \quad x \rightarrow 0.$$

(11) and (12) give

$$(13) \quad \frac{G(0)}{F(0)^a} = \lim_{x \rightarrow 0} \frac{G(x)}{F(x)^a} = \lim_{x \rightarrow 0} \frac{d + 0(x^\nu)}{(c - x^\mu u(x))^a} = \frac{d}{c^a} = 1.$$

We shall show that there is a $\varkappa > 0$ such that

$$(14) \quad \frac{G(x)}{F(x)^a} = 1 + 0(x^\varkappa), \quad x \rightarrow 0.$$

In fact, since $c > 0$, it follows from hypothesis (iv) and (11) that we may write for x sufficiently small

$$\frac{1}{F(x)} = \frac{1}{c - x^\mu u(x)} = \frac{1}{c} \left(1 + \frac{x^\mu u}{c} + \left(\frac{x^\mu u}{c} \right)^2 + \dots \right) = \frac{1}{c} + 0(x^\mu),$$

whence

$$\frac{1}{F(x)^a} = \frac{1}{c^a} + 0(x^\mu).$$

By relation (12) and the equality $d = c^a$ we get

$$\frac{G(x)}{F(x)^a} = \left(\frac{1}{c^a} + 0(x^\mu) \right) (c^a + 0(x^\nu)) = 1 + c^a 0(x^\mu) + c^{-a} 0(x^\nu) + 0(x^{\mu+\nu}),$$

which gives (14) with $\varkappa = \min(\mu, \nu) > 0$.

(14) means that there exists a constant $K > 0$ such that the inequality

$$(15) \quad \left| \frac{G(x)}{F(x)^a} - 1 \right| \leq Kx^\varkappa, \quad \varkappa > 0,$$

holds in a neighbourhood of zero.

Brom hypothesis (iii) and (C₃) we infer that there is a constant $L > 0$ such that

$$(16) \quad \left| \frac{x^{r-pa} H(x)}{F(x)^a} \right| \leq Lx^{r-pa}, \quad r-pa > 0,$$

for x belonging to a neighbourhood of zero.

At last, hypothesis (i) implies the existence of a number ϑ , $0 < \vartheta < 1$, such that

$$(17) \quad f(x) \leq \vartheta x$$

in a neighbourhood of zero.

Let $\langle 0, \delta \rangle$ be the neighbourhood of zero in which (15), (16) and (17) hold. In $\langle 0, \delta \rangle$ there are actually fulfilled the hypotheses of the mentioned theorem ([2], th. 6), and therefore equation (10) has a one-parameter family of continuous solutions in $\langle 0, \delta \rangle$.

Thus equation (L) has in $\langle 0, \delta \rangle$ a one-parameter family of solutions in the class V^a , which was to be proved.

Remark. Of course, equation (10) has also a solution $\Phi_0(x)$ continuous in $\langle 0, \delta \rangle$ and fulfilling $\Phi(0) = 0$, but then the function

$$\varphi_0(x) = x^a \Phi_0(x), \quad a = q/(p-1)$$

does not belong to the class V^a , but to the class V^{r-a} , i.e. it is in fact the solution of equation (L) found in theorem 2 above. Now we pass to case (C₂). Then equation (4) leads to

$$F(x)^a \Phi(f(x)) = G(x) \Phi(x) + H(x),$$

i.e.

$$(18) \quad \Phi(f(x)) = \frac{G(x)}{F(x)^a} \Phi(x) + \frac{H(x)}{F(x)^a}.$$

We see from (18) that under hypotheses (iii)–(iv) we cannot have $d = c^a$. For, it would imply the equality

$$\frac{H(0)}{F(0)^a} = \frac{H(0)}{c^a} = 0$$

which is impossible by (2).