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Remarks on Some Oriantor Equations

1. We first recall fundamental definitions, taken partially from [2]. If (X, ρ) is a metric space then by $\mathcal{F}(X)$ we denote the family of all subsets of X and by $\text{comp}(X)$ the set of all compact subsets of X . The set $\text{comp}(X)$ is provided with the classical Hausdorff metric ρ defined as follows: if $A, B \in \text{comp}(X)$, then $\rho(A, B) \stackrel{\text{df}}{=} \max(\max\{d(a, B) : a \in A\}, \max\{d(b, A) : b \in B\})$, where $d(c, C) \stackrel{\text{df}}{=} \inf\{\rho(x, c) : x \in C\}$ for $c \in X, C \in \text{comp}(X)$.

A mapping $F: X \rightarrow \text{comp}(X)$ is said to be upper semi-continuous if for all sequences $\{x^p\}, \{y^p\} \subset X$, the conditions: $x^p \rightarrow x^0, y^p \rightarrow y^0, x^p \in F(y^p)$ imply: $x^0 \in F(y^0)$; F is called compact if for every bounded subset A of X the closure of the set $\bigcup \{F(x) : x \in A\}$ is compact in X . The mapping F is said to be completely continuous if it is upper semi-continuous and compact.

If E is a Banach space, then we put

$$cf(E) \stackrel{\text{df}}{=} \{A \in \mathcal{F}(E) : A \neq \emptyset, A \text{ is closed, } A \text{ is convex}\},$$

and for $A, B \in \mathcal{F}(E), t \in R (= \text{the real line})$:

$$A + B \stackrel{\text{df}}{=} \{x + y : x \in A, y \in B\}, \quad tA \stackrel{\text{df}}{=} \{tx : x \in A\}.$$

We shall write $A + x^0$ in the place of $A + \{x^0\}$.

If $G, F: E \rightarrow cf(E)$ then $G \subset F$ means $G(x) \subset F(x)$ for $x \in E$, $F + G$ is the mapping from E into $cf(E)$ defined by the formula $(F + G)(x) = F(x) + G(x)$. A mapping $F: E \rightarrow cf(E)$ is called homogeneous if for every $x \in E$ and every $t \in R, F(tx) = tF(x)$.

Theorem L-O (Theorem 1 in the paper of A. Lasota and Z. Opial [2]). Let F and G be two completely continuous mappings from E into $cf(E)$, such that F is homogeneous, $G \subset F + K$, where $K: E \rightarrow cf(E)$ is a constant map defined by $x \mapsto \{y : |y| \leq r\}$, r fixed ($|\cdot|$ the norm in E), and moreover if $x \in F(x)$ then $x = 0$. Under these assumptions there exists $x \in E$ such that $x \in G(x)$.

Using the methods of A. Lasota and Z. Opial presented in [1] and [2], we shall now give some results concerning a functional-differential equation of the orientor type. A similar result is given in [3].

2. If Δ is a closed interval in R , then by $C^n(\Delta)$ we denote the space of all continuous mappings $u: \Delta \rightarrow R^n$ provided with the norm of the uniform convergence: $\|u\| = \max\{|u(t)|: t \in \Delta\}$, where $|u|$ is the Euclidean norm of u in R^n .

Let a, b, c be fixed real numbers such that $c \leq a < b$. By C we denote the set of all continuous mappings from $[a, b]$ into $[c, b]$. For a set $A \subset R^n$ we put $|A| = \sup\{|x|: x \in A\}$. If $F: R^n \rightarrow cf(R^n)$, then we put

$$\int_a^x F(t) dt \stackrel{\text{df}}{=} \left\{ \int_a^x w(t) dt: w(t) \in F(t), w \in L_{[a,b]}^1 \right\}$$

(see for instance [2]), where $L_{[a,b]}^1$ denotes the family of real summable functions on $[a, b]$.

Proposition. Let F and L be mappings from R^n into $cf(R^n)$ and from $C^n([a, b])$ into R^n respectively, and let $u \in C^n([a, b])$ and $r \in R^n$ be fixed. Then the two following conditions are equivalent, under the assumption that u is absolutely continuous:

(a) $u'(x) \in F(x)$ for almost every $x \in [a, b]$, and $Lu = r$;

(b) $u(x) \in \int_a^x F(t) dt + Lu - r + u(a)$.

The proof will be omitted (compare [1, 2]).

If (X, ϱ) and (Y, r) are two metric spaces, then in $X \times Y$ we shall consider the metric s as always defined by the formula:

$$s((x, y), (u, v)) = \varrho(x, u) + r(y, v).$$

Hence, in particular, a mapping $H: R^n \times \mathcal{A} \rightarrow cf(R^n)$ where \mathcal{A} is a family of compact subsets of R^n , is upper semi-continuous if and only if the following four conditions:

(1) $x^k \in R^n, k = 0, 1, \dots, x^k \rightarrow x^0$ as $k \rightarrow \infty$,

(2) $y^k \in R^n, k = 0, 1, \dots, y^k \rightarrow y^0$ as $k \rightarrow \infty$,

with respect to the Hausdorff metric,

(3) $A^k \in \mathcal{A}, k = 0, 1, \dots, A^k \rightarrow A^0$ as $k \rightarrow \infty$

(4) $y^k \in H(x^k, A^k), k = 1, 2, \dots$,

imply the condition:

(5) $y^0 \in H(x^0, A^0)$.

Definition 1. (see [1]). We say that a function $h: [a, b] \times R^n \times \mathcal{F}(R^n) \rightarrow cf(R^n)$ fulfils the condition of Carathéodory if and only if:

(i) for almost every $x \in [a, b]$, the mapping

(6) $R^n \times \mathcal{F}(R^n) \ni (u, A) \mapsto h(x, u, A) \in cf(R^n)$

is upper semi-continuous,

(ii) for every $(u, A) \in R^n \times \mathcal{F}(R^n)$, the mapping

$$(7) \quad [a, b] \ni x \mapsto h(x, u, A) \in cf(R)$$

is measurable.

(iii) there exist summable functions $\varrho_1, \varrho_2, \mu: [a, b] \rightarrow R$, such that

$$(8) \quad |h(x, u, A)| \leq \varrho_1(x)|u| + \varrho_2(x)|A| + \mu(x)$$

for $(x, u, A) \in [a, b] \times R^n \times \mathcal{F}(R^n)$.

Definition 2. We say that a mapping $Z: C^n([a, b]) \rightarrow cf(C^n[a, b])$ fulfils the hypothesis (H) if the following conditions (9)-(11) hold:

$$(9) \quad v \in Z(u) \Rightarrow v(a) = u(a),$$

$$(10) \quad Z \text{ is completely continuous,}$$

$$(11) \quad Z \text{ is homogeneous.}$$

3. Suppose now, that $L: C^n([a, b]) \rightarrow R^n$ is linear and continuous, $f: [a, b] \times R^n \times \mathcal{F}(R^n) \rightarrow cf(R^n)$ fulfils the Carathéodory condition, Z is a mapping fulfilling (H) (see Def. 1 and 2), and furthermore, the mapping $\Phi: [a, b] \rightarrow \text{comp}([c, b]) =$ the set of all compact subsets of $[c, b]$, is such that for every $u \in C^n([c, b])$ the set

$$(12) \quad W = W(\Phi, f; u) = \{w \in L^1_{[a,b]}: w(x) \in f(x, u(x), u(\Phi(x))) \text{ for } x \in [a, b]\}$$

is non-empty.

Remark 1. Putting

$$\Phi(x) = \{\varphi(t): \varphi \in C, \varphi(t) \in [\varphi^0(t), \varphi^1(t)] \text{ for } t \in [a, b]\},$$

where $\varphi^0, \varphi^1 \in C$ are fixed and such that $\varphi^0(t) \leq \varphi^1(t)$ for $t \in [a, b]$, we obtain an example of $\Phi: [a, b] \rightarrow \text{comp}([c, b])$ for which $W(\Phi, f; u) \neq \emptyset$ for any $u \in C^n([c, b])$ and any f fulfilling the Carathéodory condition.

We define now $F = F_{f,r,Z,\Phi,L}: C^n([c, b]) \rightarrow cf(C^n([c, b]))$, as follows:

$$(13) \quad F(u) = \{w \in C^n([c, b]):$$

$$1^\circ w|_{[a,b]}(x) \in \int_a^x f(t, u(t), u(\Phi(t))) dt + L(u|_{[a,b]}) - r + u(a) \text{ for } x \in [a, b], \text{ and}$$

$$2^\circ w|_{[c,a]} \in Z(u|_{[c,a]} + L(u|_{[a,b]}) - r\}.$$

Here $w|_{[c,a]}$ (and similar symbols) denotes the restriction of w to the set $[c, a]$; $L(u|_{[a,b]}) - r$ is considered as a constant map: $[c, a] \ni x \mapsto L(u|_{[a,b]}) - r \in R^n$.

From the assumption of the convexity of Z and f , it follows, that $F(u)$ is really a convex subset of $C^n([c, b])$ for every u . The closedness of $F(u)$ is also obvious. Note that if $w \in F(u)$, then $w(a) = u(a) + L(u|_{[a,b]}) - r$.

Lemma. If f, Z, Φ, L are as above, then F defined by (13) is for every $r \in L(C^n([a, b]))$ completely continuous.

Proof. In order to prove that F is upper semi-continuous we apply the reasoning given in [1], without any essential changes. Let $\{u^p\}$ and $\{z^p\}$ ($C^0([c, b])$) be convergent uniformly to u^0 and z^0 respectively, i.e.: $\|u^p - u^0\|, \|z^p - z^0\| \rightarrow 0$ as $p \rightarrow \infty$. Let $z^p \in F(u^p)$ for $p = 1, 2, \dots$. From the assumptions it directly follows that $z^0|_{[c, a]} \in Z(u^0|_{[c, a]} + L(u^0|_{[a, b]} - r)$. Moreover, there exists a sequence $\{v^p\}_{p=1, 2, \dots} \subset (L^1_{[a, b]})^n$, such that $z^p(x) = \int_a^x v^p(t) dt + Lu^p - r + u^p(a)$ and

$$v^p(x) \in f(x, u^p(x), u^p(\Phi(x))) \text{ for } p = 1, 2, \dots, \quad x \in [a, b].$$

Since $\{u^p\}$ converges uniformly to u^0 in $[a, b]$ and f fulfils the Carathéodory condition, there exists a function $\bar{v} \in L^1_{[a, b]}$, such that $\|v^p(x)\| \leq \bar{v}(x)$ almost everywhere in $[a, b]$. By Lemma 2 from the paper [1], there exists a double sequence $\{\lambda_{ij}\}$ ($i = 1, 2, \dots, j = i, i+1, \dots$) of real non-negative numbers, such that $\sum_{j=1}^{\infty} \lambda_{ij} = 1$, $\lambda_{ij} = 0$ for sufficiently large j (depending on i), and the sequence $\tilde{v}^i = \sum_{j=1}^{\infty} \lambda_{ij} v^j$ ($i = 1, 2, \dots$) converges almost everywhere in $[a, b]$ to a function $v^0 \in (L^1_{[a, b]})^n$. We have $\tilde{v}^i(x) = \sum_{j=1}^{\infty} \lambda_{ij} v^j(x) \rightarrow v^0(x)$ almost everywhere in $[a, b]$, and then

$$\int_a^x \tilde{v}^i(t) dt = \sum_{j=1}^{\infty} \lambda_{ij} \int_a^x v^j(t) dt \rightarrow \int_a^x v^0(t) dt \text{ as } i \rightarrow \infty.$$

We have

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_{ij} z^j(x) &= \sum_{j=1}^{\infty} \lambda_{ij} \int_a^x v^j(t) dt + \sum_{j=1}^{\infty} \lambda_{ij} L(u^j) - r + \sum_{j=1}^{\infty} \lambda_{ij} u^j(a), \text{ and then } z^0(x) \\ &= \int_a^x v^0(t) dt + L(u^0) - r + u^0(a). \end{aligned}$$

From the uniform convergence of $\{u^p\}$ to u^0 , we obtain $u^p(\Phi(x)) \rightarrow u^0(\Phi(x))$ in the sense of the Hausdorff metric. In virtue of the upper semi-continuity of f we finally obtain $v^0 \in f(x, u^0(x), u^0(\Phi(x)))$, which means that $z^0 \in F(u^0)$. Let now A be a bounded subset of $C^0([c, a])$. Consider the closure of $\bigcup \{F(u) : u \in A\}$ and denote it by $F[A]$. Let $\{z^p\}$ be the sequence of elements belonging to $F[A]$. Directly from the assumptions we have the compactness of the closure of the set: $\bigcup \{Z(u|_{[c, a]} + Lu - r) : u \in A\}$ (here $Lu = L(u|_{[a, b]})$). Denote this closure by $Z[A; L, r]$. Hence we can assume that $\{z^p|_{[c, a]}\}$ converges uniformly to a function $\tilde{z} \in Z[A; L, r]$.

There exist sequences $\{u^p\} \subset A$ and $\{v^p\} \subset (L^1_{[a, b]})^n$, such that $z^p(x) = \int_a^x v^p(t) dt + Lu^p - r + u^p(a)$, $v^p(x) \in f(x, u^p(x), u^p(\Phi(x)))$, for $x \in [a, b]$, $p = 1, 2, \dots$.

From the Carathéodory condition (see (iii)) it follows that

$$\left| \int_a^x v^p(t) dt \right| \leq \int_a^x (\varrho_1(t)|u^p(t)| + \varrho_2(t)|u^p(\Phi(t))| + \mu(t)) dt$$

