ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCLVI PRACE MATEMATYCZNE, ZESZYT 16 1974

*Stanisław Sędziwy*

# On the Existence of a Solution of a Certain Nonlinear Boundary Value Problem

1. This note concerns the existence of solutions to the boundary value problem (BVP for short)

$$x'' = f(t, x, x'), \tag{1}$$

$$(x(0), x'(0)) \in S, \quad (x(1), x'(1)) \in C, \tag{2}$$

where $f\colon I \times R^2 \to R$ is continuous ($I$ denotes the closed unit interval of the real line $R$) and sets $C, S \subset R^2$ are connected, moreover $S$ is compact and $C$ is closed.

Recently Wilhelmsen [6], Bebernes and Wilhelmsen [1], [2], Jackson and Klaasen [4] have given sufficient conditions for the existence of solutions to BVP (1), (2). One of these conditions consists in assuming the existence of a strict lower and a strict upper solution of (1), i.e. functions $a(t), b(t) \in C^2(I)$, satisfying on $I$ the inequalities

$$a''(t) > f(t, a(t), a'(t)), \quad b''(t) < f(t, b(t), b'(t)).$$

(Note that conditions assumed in [6] imply the existence of strict upper and lower solutions of (1) having the properties imposed in [1], [2] or [4]).

The purpose of this note is to show that the results of the mentioned papers remain valid if the existence of lower and upper solutions of (1) is assumed.

Recall that $a(t), b(t) \in C^2(T)$ are said to be a lower solution of (1) and an upper solution of (1) on $I$ if

$$a''(t) \geqslant f(t, a(t), a'(t)), \quad b''(t) \leqslant f(t, b(t), b'(t)) \tag{3}$$

holds for all $t \in I$.

2. Theorem. Assume that

(A) There exist functions $a(t), b(t) \in C^2(I)$, satisfying (3) and such that $a(t) \leqslant b(t)$ on $I$.

(B) Given any $N > 0$, $\tau_0 \in I$, there exists $M(N) > 0$ such that for any solution $x = x(t)$ of (1) defined on $[0, \tau_0] \subset I$ with $a(t) \leqslant x(t) \leqslant b(t)$, $|x'(0)| \leqslant N$ implies that $|x'(t)| \leqslant M(N)$ on $[0, \tau_0]$.

If $S$ and $C$ satisfy

$$S \subset [a(0), b(0)] \times R,$$
(4)
$$S \cap (\{a(0)\} \times (-\infty, a'(0)]) \neq \emptyset, \quad S \cap (\{b(0)\} \times [b'(0), \infty)) \neq \emptyset,$$

(for $x_0 \in R$, $\{x_0\}$ denotes a one-point subset of $R$),

(5) $C \cap (R \times \{p\}) \neq \emptyset$ and $C \cap (R \times \{p\}) \subset ([a(1), b(1)] \times \{p\})$ for all $p \in R$,

then BVP (1), (2) has at least one solution $x = x(t)$ with $a(t) \leqslant x(t) \leqslant b(t)$ on $I$.

Put $D = \{(t, x, y): a(t) \leqslant x \leqslant b(t), (t, y) \in I \times R\}$ and define $g: I \times R^2 \to R$ by

$$g(t, x, y) = \begin{cases} f(t, x, y) & \text{for } a(t) \leqslant x \leqslant b(t), \\ f(t, a(t), y) + (x - a(t)) & \text{for } x \leqslant a(t), \\ f(t, b(t), y) + (x - b(t)) & \text{for } x \geqslant b(t). \end{cases}$$

To prove the theorem it is sufficient to show the existence of a solution of the equation

(6) $$x'' = g(t, x, x')$$

satisfying (2) and the condition

(7) $$(t, x(t), x'(t)) \in D \quad \text{for } t \in I.$$

Before proceeding to the proof of the existence of such solution, we shall prove Lemma.

3. Lemma. Let $t_0 \in I$, $b(t_0) \leqslant x_0$, $b'(t_0) \leqslant y_0$. Let all solutions of (6) through $(t_0, x_0, y_0)$ exist on $[t_0, s] \subset I$. Then there is a solution $x = x(t)$ of (6), $x_0 = x(t_0)$, $y_0 = x'(t_0)$ such that $b(s) \leqslant x(s)$, $b'(s) \leqslant x'(s)$.

Proof. If $(x_0, y_0) \neq (b(t_0), b'(t_0))$ the assertion of Lemma follows from the inequality $b''(t) \leqslant g(t, b(t), b'(t))$ and the differential inequality theorem ([3], Ex. 4.1, 4.3, p. 28).

Let $(x_0, y_0) = (b(t_0), b'(t_0))$. Let $x = x_n(t)$ $(n = 1, 2, \ldots)$ be solutions of (6), $x(t_0) = x_0$, $x_n'(t_0) = y_n$, where $y_n > y_0$ and $\lim_{n\to\infty} y_n = y_0$. For $n$ large enough all $x_n(t)$ are defined on $[t_0, s]$. Moreover there is a subsequence $\{x_{k(n)}(t)\}$ of $\{x_n(t)\}$ such that $\lim_{n\to\infty} x_{k(n)}(t) = x(t)$ uniformly on $[t_0, s]$, where $x(t)$ is a solution of (6) satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = y_0$ ([3], Th. 2.4, p. 4). By the first part of the proof, $x_{k(n)}(s) \geqslant b(s)$, $x'_{k(n)}(s) \geqslant b'(s)$. Thus $x(s) \geqslant b(s)$, $x'(s) \geqslant b'(s)$ which completes the proof of Lemma.

Remark. A similar assertion (with reversed inequalities) can be proved for $a(t)$.

4. Proof of Theorem. Replace (6), (7) by

(8) $$x' = y, \quad y' = g(t, x, y),$$

(9) $$(t, x(t), y(t)) \in D \quad \text{for } t \in I.$$

For $(t_0, x_0, y_0) \in D$, $t_1 \in I$ define the set $I(x_0, y_0, t_0, t_1)$ by $I(x_0, y_0, t_0, t_1) = \{(t_1, u, w)\colon u = x(t_1),\ w = y(t_1),\ x(t), y(t)$ satisfy (8) and $x(t_0) = x_0,\ y(t_0) = y_0\}$.

If $\{t_0\} \times A \subset D$ let $I(A, t_0, t_1) = \cup \{I(x_0, y_0, t_0, t_1)\colon (x_0, y_0) \in A\}$. Moreover define in $I \times R^2$ the sets $K_1, K_2$: $K_1 = \{(t, x, y)\colon b(t) \leqslant x,\ b'(t) \leqslant y,\ t \in I\}$, $K_2 = \{(t, x, y)\colon x \leqslant a(t),\ y \leqslant a'(t),\ t \in I\}$.

Let $N = \max\{|y|\colon (x, y) \in S\}$, $\tau_0 = 1$ and let $M(N)$ be chosen according to (B). Assume that $n_0$ is so large that all solutions of (8) with initial points $(t_0, x_0, y_0) \in D \cap \{(t, x, y)\colon |y| \leqslant M(N)\}$ are defined for $|t - t_0| \leqslant 2^{-n_0}$.

Let $t_i^m = i/2^n$ $(i = 0, 1, \ldots, m;\ m = 2^n)$. For $n \geqslant n_0$ denote by $S_i^m$ $(i = 1, 2, \ldots, m;\ m = 2^n)$ a compact connected component of $I(S_{i-1}^m, t_{i-1}^m, t_i^m) \cap D$ which intersects $K_1$ and $K_2$, $S_0^m = \{0\} \times S$.

By (4), $S_0^m \subset D$ and $S_0^m \cap K_j \neq \emptyset$ $(j = 1, 2)$. The compactness and connectedness of $S_0^m$ imply that $I(S_0^m, 0, t_1^m)$ is also compact and connected (see for example [5]). By Lemma and Remark, $I(S_0^m, 0, t_1^m) \cap K_j \neq \emptyset$, which implies the existence of $S_1^m$, which is obviously compact. (B) implies that $S_1^m \subset D \cap \{(t, x, y)\colon |y| \leqslant M(N)\}$, hence, by induction, we conclude that $S_i^m$ are defined for all $i$.

Let $C_0 = \{1\} \times C$. It is clear from (5) that $C_0 \cap S_m^m \neq \emptyset$. Since $S_m^m \subset I(S_0^m, 0, 1)$, this implies that there is a solution $x = x_m(t)$, $y = y_m(t)$ of (8) satisfying

$$(10) \qquad (t_i^m, x_m(t_i^m), y_m(t_i^m)) \in D \quad (i = 1, 2, \ldots, m), \qquad (1, x_m(1), y_m(1)) \in C_0 .$$

From the definition of sets $S_i^m$ it follows that there exists an open set $G$ having compact closure and containing $D$ such that $(t, x_m(t), y_m(t)) \in G$ for all $t \in I$, $m = 2^n$, $n \geqslant n_0$. Hence the family $\{x_m(t), y_m(t)\}$ is uniformly bounded. $\max\{|y| + |g(t, x, y)|\colon (t, x, y) \in G\} < \infty$ implies that it is equicontinuous. By the Ascoli Theorem, there is a subsequence of $\{x_m(t), y_m(t)\}$ converging uniformly on $J$ to the solution $x = x(t)$, $y = y(t)$ of (8). By (10) and the compactness of $D$, $x(t), y(t)$ satisfies (9). From (10) and the closedness of $C_0$ it follows that $(x(1), y(1)) \in C$, which ends the proof of Theorem.

## REFERENCES


[1] J. W. Bebernes, R. Wilhelmsen, *A technique for solving two dimensional boundary-value problems*, SIAM Journ. Appl. Math., 17 (1969), 1060—1064.

[2] —, —, *A general boundary value problem technique*, J. Diff. Eqs., 8 (1970), 404—415.

[3] P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

[4] L. K. Jackson, G. Klaasen, *A variation of the topological method of Ważewski*, SIAM J. Appl. Math., 20 (1971), 124—130.

[5] T. Ważewski, *On an optimal control problem, Diff. equations and their applications*, Proc. Conf. Prague, Sept. 1962, 233.

[6] R. Wilhelmsen, *A nonlinear boundary-value problem*, Bull. AMS, 73 (1967), 920—921.