ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCLVI PRACE MATEMATYCZNE, ZESZYT 16 1974

Kamil Rusek

Remarks on H-bounded Subsets in Banach Space

The aim of this paper is to give a generalization and simple proofs of some of Dineen's results ([3], [4]) characterizing the subsets of Banach space on which every entire function is bounded.

Let E be a Banach space over the field $\mathbf{K}$ (equal $\mathbf{R}$ or $\mathbf{C}$), E' be the dual space and let $B_1(E) = \{x \in E: \|x\| \leqslant 1\}$.

We denote by $P^k(E)$ the Banach space of all continuous homogeneous polynomials from E into $\mathbf{K}$ of degree k. (We set $P^0(E) = \mathbf{K}$).

Definition 1. A function f: $E \to \mathbf{K}$ is called *entire*, if there exists a sequence $\{f_n: f_n \in P^n(E),\ n = 0, 1, \ldots\}$ such that the series $\sum_0^\infty f_n(x)$ is convergent for every $x \in E$ and

$$f(x) = \sum_0^\infty f_n(x),\quad x \in E.$$

Every entire function is analytic on E (see [2], Th. 5.2.)

The vector space of all entire functions from E into $\mathbf{K}$ is denoted by $H(E)$. We define

$$H_k(E) = \{f \in H(E): f = \sum_1^\infty g_n^n \text{ for some } \{g_n\}_1^\infty \subset P^k(E)\}$$

for $k = 1, 2, \ldots$

The set $H_k(E)$ is completely described by the following

Proposition 1 (see [4], when $\mathbf{K} = \mathbf{C}$). *Let* $\{g_n\}_1^\infty \subset P^k(E)$. *Then*

$$f = \sum_1^\infty g_n^n \in H(E) \Leftrightarrow \forall x \in E: g_n(x) \to 0.$$

Proof. ($\Rightarrow$). Fix an $x \in E$. Then, for every $t \in \mathbf{K}$, the series $\sum_{n=1}^\infty t^{kn} g_n^n(x)$ is convergent. Therefore, $g_n(x) \to 0$.

($\Leftarrow$). Since $g_n^n \in P^{kn}(E)$ for $n=1,2,\ldots$ and since, for every $x \in E$, the series $\sum_1^\infty g_n^n(x)$ is convergent, the function $f=\sum_1^\infty g_n^n$ is entire.

Now we give two basic definitions.

Definition 2. A subset X of E is called *$H(E)$-bounded ($H_k(E)$-bounded)*, if for every $f \in H(E)$ (for every $f \in H_k(E)$) $\|f\|_X \stackrel{\text{df}}{=} \sup\{|f(x)|: x \in X\} < \infty$.

It is obvious that every $H(E)$-bounded subset (or $H_1(E)$-bounded) is bounded and the set $\bar{X}$ is $H(E)$-bounded ($H_1(E)$-bounded) too.

Condition (G_k). We say that $X \subset E$ satisfies the condition (G_k) (we write: $X \in (G_k)$) if for every $\{g_n\}_1^\infty \subset P^k(E)$, $g_n(x) \to 0$ for every $x \in E$ implies $\|g_n\|_X \to 0$.

Condition (G_∞). $X \subset E$ satisfies the condition (G_∞) (we write: $X \in (G_\infty)$) if for every $\{g_n: g_n \in P^n(E),\ n=0,1,\ldots\}$, $|g_n(x)|^{1/n} \to 0$ for every $x \in E$ implies $\|g_n\|_X^{1/n} \to 0$.

Condition (G_k) is explained by the following theorem (see [7]).

Theorem 1 (Gelfand). *Let E be a Banach space and let $X=\bar{X} \subset E$ be bounded. If*

(*) *every bounded sequence in E' has a weak * convergent subsequence*

then

X is compact $\Leftrightarrow X \in (G_1)$.

Remark 1. Gelfand's theorem is formulated in [7] for Banach spaces over **R** without the assumption (*), which, however, is used in the proof. Actually, assumption (*) cannot be omitted.

We obtain the complex case by the natural isomorphism between $(E_{\mathbf{R}})'$ and $Re E'=\{Re f: f \in E'\}$ ($E_{\mathbf{R}}$ is the space E treated as a vector space over **R**).

Every separable and every reflexive Banach space has the property (*).

The space l_∞ does not have the property (*) (cf. Remark 2).

There is a connection between Definition 2 and conditions (G_k) and (G_∞); namely we have (about 2° see [3]).

Lemma 1. 1° *$X \subset E$ is $H_k(E)$-bounded $\Leftrightarrow X \in (G_k)$.*

2° *$X \subset E$ is $H(E)$-bounded $\Leftrightarrow X \in (G_\infty)$.*

Proof. 1° ($\Leftarrow$). By Prop. 1.

($\Rightarrow$). Suppose that the set X does not satisfy condition (G_k). Then there exists $\{f_n\}_0^\infty \subset P^k(E)$ and $\{x_n\}_0^\infty \subset X$ such that $f_n(x) \to 0$ for every $x \in E$ and $|f_n(x_n)| \geq 2$ for $n \geq 0$.

Put $n_0=1$. Take an n_1 such that $|f_s(x_{n_0})| \leq 2^{-1}$ for $s \geq n_1$ and $|f_{n_1}^{n_1}(x_{n_1})| \geq 2^{n_1} \geq 2+|f_{n_0}^{n_0}(x_{n_1})|$.

By iteration we obtain the sequence of positive integers $1=n_0<n_1<n_2<\ldots$ such that

$$|f_s(x_{n_{l-1}})| \leq 2^{-1},\ s \geq n_l, \tag{1}$$

$$|f_{n_l}^{n_l}(x_{n_l})| \geq 2^l+\Big|\sum_{j=1}^{l-1} f_{n_j}^{n_j}(x_{n_l})\Big|. \tag{2}$$

Put $f=\sum_{l=0}^{\infty} f_{n_l}^{m_l}$. Then $f\in H_k(E)$ and by (1) and (2)

$$|f(x_{n_l})| \geq |f_{n_l}^{m_l}(x_{n_l})| - \Big|\sum_{j=0}^{l-1} f_{n_j}^{m_j}(x_{n_l})\Big| - \Big|\sum_{j=l+1}^{\infty} f_{n_j}^{m_j}(x_{n_l})\Big| \geq 2^l - \sum_{j=l+1}^{\infty} (2^{-j})^{m_j}$$

$$\geq 2^l - \sum_{j=0}^{\infty} 2^{-j} = 2^l - 2\,.$$

Thus, $\|f\|_X = \infty$ and 1° is proved.

Ad 2° ($\Leftarrow$) — obvious.

($\Rightarrow$). If X does not satisfy G_∞, one find a sequence of integers $n_0 < n_1 < \ldots$, a sequence of polynomials $f_{n_k} \in P^{n_k}(E, K)$, $k \geq 0$, and a sequence of points $\{x_k\} \subset X$ such that

$$|f_{n_k}(x_k)| \geq 2^{n_k} (k \geq 0) \text{ and } \lim_{k\to\infty} |f_{n_k}(x)|^{\frac{1}{n_k}} = 0 \quad \text{for } x\in E\,.$$

Now, analogously as in 1°, one may find a sequence of integers $1 = k_0 < k_1 < \ldots$ such that for $l = 1, 2, \ldots$

$$|f_{n_s}(x_{k_{l-1}})|^{\frac{1}{n_s}} < 2^{-1},\ s \geq k_l \tag{1}$$

$$|f_{n_{k_l}}(x_{k_l})| > 2^l + \Big|\sum_{j=0}^{l-1} f_{n_{k_j}}(x_{k_l})\Big|\,. \tag{2}$$

The function $f = \sum_{l=0}^{\infty} f_{n_{k_l}}$ is entire and it is not bounded on X.

The following theorem is a slight generalization of a Dineen's result (see [4], Prop. 3).

Theorem 2. *Let E satisfy assumption $(*)$ and let $X = \overline{X} \subset E$. Then the following statements are equivalent:*

(i) *X is $H(E)$-bounded,*

(ii) *X is $H_k(E)$-bounded for every $k \geq 1$,*

(iii) *$X \in (G_k)$ for every $k \geq 1$,*

(iv) *$X \in (G_k)$ for some $k \geq 1$,*

(v) *$X \in (G_1)$,*

(vi) *$X \in (G_\infty)$,*

(vii) *X is $H_1(E)$-bounded,*

(viii) *X is compact.*

Proof. By Lemma 1 and Theorem 1.

Remark 2. In [4] it is proved, that (i)$\Leftrightarrow$(vii)$\Leftrightarrow$(viii).

(i)$\Rightarrow$(viii) does not hold true when assumption $(*)$ is omited. For instance, Dineen proved in [3] that in the complex space l_∞ the set $X = \{e_n\colon e_n = (0, \ldots, 0, \overset{(n)}{1}, 0, \ldots),\ n = 1, 2, \ldots\}$ is $H(l_\infty)$-bounded, though it is not compact.

The author does not know the answer to the following question.

Problem 1. Does (v)$\Rightarrow$(iii)$\Rightarrow$(i) remain true for an arbitrary Banach space? It would be of interest to give a reply to Problem 1 even for l_∞.

In the study of analytic continuation one meets the following open.

Problem 2 (see [3], [4], [5]). Does there exist a Banach space E, $\dim E = \infty$, such that $B_1(E)$ is $H(E)$-bounded?

Problem 2'. Does there exist a Banach space E, $\dim E = \infty$, such that $B_1(E) \in (G_1)$?

Relations between Problems 1, 2 and 2' are given by Theorem 2.

Remark 3. Dineen proved in [4] that $B_1(l_\infty)$ and $B_1(l'_\infty)$ do not satisfy condition (G_1).

Remark 4. It follows from Phillips theorem (see [6], p. 14) that $B_1(c_0) \subset l_\infty$ satisfies condition (G_1), i.e. $B_1(c_0)$ is a $H_1(l_\infty)$-bounded subset of l_∞.

There is Hirschowitz's conjecture (see [5]) that $B_1(l_\infty/c_0)$ is $H(l_\infty/c_0)$-bounded subset of l_∞/c_0.

The negative reply to this conjecture is due to Dineen in [3].

Now we give an immediate simple proof of this fact.

Proposition 2. $B_1(l_\infty/c_0) \notin (G_1)$.

Proof. By a factor space property (see [1], p. 152)

$$(l_\infty/c_0)' \simeq c_0^\perp, \text{ where } c_0^\perp = \{f \in l'_\infty \colon c_0 \subset f^{-1}(0)\}.$$

Therefore, it is enough to construct the sequence $\{h_n\}_1^\infty \subset c_0^\perp$ such that $h_n \to 0$ in the weak * topology for l'_∞ and $\|h_n\| \not\to 0$.

For every Banach space E

$$E''' = I'(E') \oplus I(E)^\perp$$

where $I\colon E \to E''$ and $I'\colon E' \to E'''$ are the natural embeddings (see [6], p. 73).

Putting $E = c_0$ we obtain

$$l'_\infty = l_1 \oplus c_0^\perp, \tag{3}$$

where l_1 and c_0 are isometrically embedded into l'_∞ and l_∞, respectively.

By Remark 3 there exists a sequence $\{f_n\}_1^\infty$ in l'_∞ such that $f_n \to 0$ in the weak * topology, but $\|f_n\| \not\to 0$. By (3), for every $n \geq 1$ there exist a $g_n \in l_1$ and an $h_n \in c_0^\perp$ such that

$$f_n = g_n + h_n. \tag{4}$$

We shall show that $h_n \to 0$ in the weak * topology in l'_∞ while $\|h_n\| \not\to 0$.

Since $h_n = 0$ on c_0, we have

$$\|f_n\|_{B_1(c_0)} = \|g_n\|_{B_1(c_0)} = \|g_n\| \tag{5}$$

($l_1 = c'_0$ is isometrically embedded into l'_∞).

By Remark 4 $g_n \to 0$ in the norm topology in l'_∞ and, consequently, in the weak * topology. By (4), $h_n \to 0$ in the weak * topology. Since $\|f_n\| \leqslant \|g_n\| + \|h_n\|$ and $\|g_n\| \to 0$, we have $\|h_n\| \nrightarrow 0$.

Therefore $\{h_n\}_1^\infty$ has the required properties and the proof is completed.

I would like to thank Professor Józef Siciak for helpful discussions on this problem.

REFERENCES

[1] A. Aleksiewicz, *Analiza funkcjonalna*, Warszawa 1969.

[2] J. Bochnak, and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math., 39 (1971).

[3] S. Dineen, *Bouding subsets of a Banach space*, Math. Ann. 192 (1971).

[4] —, *Unbounded holomorphic functions on a Banach space*, J. London Math. Soc. (2), 4 (1972).

[5] A. Hirschowitz, *Bornologie des espaces de fonctions analytiques en dimension infinie*, Sém. Lelong 70, Springer Lecture Notes, 205 (1971).

[6] T. Marti Jürg, *Introduction to the theory of bases*, Berlin—Heidelberg—New York 1969.

[7] W. D. Milman, *Geometriczeskaja tieoria prostranstw Banacha*, cz. I, Uspiechi Mat. Nauk, 25, 3 (1970).