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Criteria for Analyticity in Topological Vector Spaces

1. Introduction. Let E be a metrizable topological vector space and U be an open subset of E. Let F be a locally convex, sequentially complete space with the conjugate space F' (all vector spaces in this paper are complex).

Regarding the definitions and properties of analytic and G-analytic mappings between topological vector spaces we refer to [2].

If $M \subset F'$ we introduce the following

Definition 1. A function f: $U \to F$ is called *$^*M$-analytic* if for every $\varphi \in M$, he functio n $\varphi \circ f$ is analytic.

If $M = F'$, an $^*F'$-analytic function is called *weakly analytic*.

It is well known (see [2], Th. 6.3) that f: $U \to F$ is analytic if and only if it is weakly analytic.

In relation to this theorem the following question arises: when, for $M \neq F'$, is every $^*M$-analytic function analytic?

If E and F are Banach spaces, F has a Schauder basis $\{e_j\}_1^\infty$, $M = \left\{e_j^*\colon x = \sum_1^\infty x_i e_i \to x_j\right\}$ and f is $^*M$-analytic, some sufficient conditions that f be analytic are due to Aron and Cima in [1] (cf. Th. 3).

Using the results of [2] we shall give a simple proof of a more general theorem.

2. Criteria for analyticity. Let F be a Banach space.

Definition 2. A closed subspace $M \subset F'$ is *determining* for F if

$$\sup\{|\varphi(x)|\colon \varphi \in M, \|\varphi\| \leqslant 1\} = \|x\|$$

for every $x \in F$.

The following theorem is a slight generalization of the well-known Dunford theorem (see [3]).

Theorem 1. *Assume that $M \subset F'$ is such that $\overline{\text{lin}\, M}$ (lin M denotes the linear envelope of M) is determining for F, $\Omega \subset \mathbf{C}$ is open and f: $\Omega \to F$ is $^*M$-analytic.*

Then

f is analytic $\Leftrightarrow f$ is semi-bounded, i.e. for every compact set $K \subset \Omega$, $f(K)$ is bounded.

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) By the Dunford theorem it is enough to show that $\psi \circ f$ is analytic for every $\psi \in \overline{\text{lin}\, M}$. Let $\psi \in \overline{\text{lin}\, M}$. Then $\psi \circ f(x) = \lim_{n\to\infty} \psi_n \circ f(x)$ for $x \in \Omega$ and, by the assumption, $\{\psi \circ f\}_1^\infty$ is convergent to $\psi \circ f$ uniformly on every compact subset of Ω. By the Waierstrass theorem, $\psi \circ f$ is analytic.

Remark 1. The assumption of semi-boundedness of f cannot be omitted. A suitable counterexample is given in [1].

The main result of this note is the following:

Theorem 2: *Let $M \subset F'$ and $\overline{\text{lin}\, M}$ be determining for F. If f: $U \to F$ is $^*M$-analytic then the following conditions are equivalent*:

(i) *f is analytic,*

(ii) *f is continuous,*

(iii) *f is compact (i.e. $f(K)$ is compact, if $K \subset U$ is compact),*

(iv) *f is semi-bounded,*

(v) *f is weakly continuous,*

(vi) *f is locally bounded.*

Proof. The implications (i)$\Rightarrow$ (ii)$\Rightarrow$ (iii)$\Rightarrow$ (iv) are obvious.

(iv)$\Rightarrow$ (i). By Th. 6.1 in [2] it is enough to show that f is G-analytic and locally bounded.

By Th. 1, f is analytic on affine lines. Hence, f is G-analytic (see [2], Prop. 5.5).

It remains to prove that f is locally bounded in U. If this is not true, there exists a $x_0 \in U$ and a sequence $\{x_n\}_1^\infty \subset U$ such that $x_n \to x_0$ while $\|f(x_n)\| > n$ which contradicts (iv).

Thus f is analytic.

(i)$\Rightarrow$ (v)$\Rightarrow$ (vi) are obvious.

(vi)$\Rightarrow$ (i). By Th. 1, f is G-analytic and, by Th. 6.1 in [2], f is analytic.

The proof is completed.

To illustrate Theorem 2 we give the following

Example. Let X be a compact Hausdorff space and let $C(X)$ be the Banach space (under the sup-norm topology) of all complex valued continuous functions on X.

Assume that f: $U \to C(X)$ is $^*M$-analytic, where

$$M = \{C(X) \ni x \to x(t) \in \mathbf{C}:\ t \in X\}.$$

Then f is analytic if and only if one of the conditions (i)-(vi) is satisfied.

Remark 2. Theorem 1 and 2 can be extended to the case when F is a locally convex complete space topologized by a family of seminorms $\Gamma(F)$ provided that the determining sets are defined as follows: $M \subset F'$ is determining if for every $q \in \Gamma(F)$, $M \cap q^{-1}(0)^0$ ($q^{-1}(0)^0$ is the polar $q^{-1}(0)$ in F') is determining for $F/q^{-1}(0)$.

3. A case when F has a Schauder basis. Now we assume that E and F are Banach spaces, F has a Schauder basis $\{e_j\}_1^\infty$ and $U \subset E$ is open.

Definition 3. We say that f: $U \to F(f(x) = \sum_1^\infty g_j(x)e_j)$ is *normal* if for every compact set $K \subset U$

$$\sup\left\{\left\|\sum_{j=n}^{\infty} g_j(x)e_j\right\| : x \in K\right\} \to 0 \text{ as } n \to \infty .$$

Theorem 2 implies the following.

Theorem 3 (see [1]). *Let* f: $U \to F$. *Assume that* $e_j^* \circ f$ *are analytic. Then the conditions* (*i*)-(vi) *in Theorem* 2 *and the condition*

(vii) f *is normal,*

are equivalent.

Proof. It follows from the Hahn–Banach Theorem that $\mathrm{lin}\{e_j^*\}$ is determining for F.

The implication (vii)$\Rightarrow$ (v) is a simple consequence of Definition 3.

By Theorem 2 it is enough to prove the implication: (iii)$\Rightarrow$ (vii).

By the Schauder basis property every mapping

Φ_n: $F \ni y \to \sum_{j=n}^{\infty} y_j e_j \in F$ is continuous and $\Phi_n(y) \to 0$ for every $y \in F$ (see [4]). Because F is a Banach space, $\Phi_n \to 0$ uniformly on compact subset of F. If $f = \{g_j\}_1^\infty$, is compact then for every compact subset $K \subset U$ we obtain

$$\sup\left\{\left\|\sum_{n}^{\infty} g_j(x)e_j\right\| : x \in K\right\} \to 0 \text{ as } n \to \infty .$$

Thus f is analytic and the proof is concluded.

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REFERENCES

[1] R. Aron, and J. Cima, *A theorem of holomorphic mappings into Banach spaces with basis*, Proc. Amer. Math. Soc. 36 (1972), 289—292.

[2] J. Bochnak and J. Siciak, *Analytic functions in topological vector spaces*, Studia Math. 39 (1972), 77—112.

[3] N. Dunford, *Uniformity in linear spaces*, Trans. Amer. Math. Soc. 44 (1938), 305—356.

[4] J. Singer, *Bases in Banach spaces*, I, Springer-Verlag, Berlin and New York 1970.