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Controllability of Linear Systems with a Stochastic Control

1. INTRODUCTION

We shall consider in this paper a system of linear differential equations with a stochastic control which admits only positive values. We shall show under the assumption of local controllability at the origin that the set of the values of the solution of this system is dense with probability one in a neighbourhood of the origin.

The necessary and sufficient condition for local controllability of linear systems with positive controls is due to S. H. Saperstone and J. Yorke ([3]).

The theorem which we shall prove is an analogue of the theorem on complete controllability proved in [2] by A. Lasota and A. Moro for a class of stochastic controls acting in an arbitrary direction.

2. DEFINITIONS AND NOTATIONS

Consider the system of linear differential equations in the real Euclidean space R^d

$$\dot{x} = Ax + bu \tag{1}$$

where A is a constant $d \times d$ matrix, b is a constant d — vector and $u(\cdot)$ is an integrable function of the real variable t with values in a certain set $\Omega \subset R$.

The system (1) is called locally controllable at the origin if there exists a finite time t_1 such that the reachable set $K_\Omega^+(t_1)$ at time t_1 contains a neighbourhood of the origin.

Let us recall that the set $K_\Omega^+(t)$ is defined by

$$K_\Omega^+(t) = \{x(t, u(\cdot)), u \in U_\Omega\}$$

where U_Ω is the set of all integrable functions $u(\cdot)$, $u\colon R^+ \cup \{0\} \to \Omega$. For each $u \in U_\Omega$ let $x(t) = x(t, u(\cdot))$ be the unique absolutely continuous function satisfying (1) such that $x(0, u(\cdot)) = 0$.

We assume in the sequel that $\Omega = [0, \delta]$ for $\delta > 0$.

It is known (see [3]) that the system (1) is locally controllable ($u \in U_{[0,\delta]}$) in a finite time t_+ at the origin if and only if

(i) all eigenvalues of A have non-zero imaginary parts and

(ii) the controllability matrix for (1) has rank d.

Consider also in the same space the system

$$\dot{x} = Ax + b\xi'(t) \tag{2}$$

and the homogeneous system related to (2)

$$\dot{x} = Ax \tag{3}$$

with the initial condition

$$x(0) = 0 \tag{4}$$

where $\xi(t)$ is a sample function of a certain random process $(\xi_t, t \geqslant 0)$.

Let us assume that

(I) $(\xi_t, t \geqslant 0)$ is a one-dimensional separable stochastic process with stationary independent increments and $E(|\xi_t|)$ is a locally bounded function of t.

In this case the characteristic function $\Phi_t(\mu)$ of the process (ξ_t) is given by Levy's formula

$$\log \Phi_t(\mu) = t\left[i\gamma\mu + \int_{-\infty}^{\infty} \left(e^{i\mu\lambda} - 1 - \frac{i\mu\lambda}{1+\lambda^2}\right) \frac{1+\lambda^2}{\lambda^2}\, dG(\lambda)\right] \tag{5}$$

where γ is a real constant, $G(\lambda)$ is a nondecreasing bounded function.

Moreover, let us assume that

(II) $(\alpha) G(\lambda) = 0$ for $\lambda \leqslant 0$
$(\beta) G(\lambda) > 0$ for $\lambda > 0$
$(\gamma) \lim G(\lambda) = 0$.

This means that the steps of the sample functions of the stochastic process are only positive ((α)) with probability one and as small ((β), (γ)) as we want with a positive probability.

3. RESULTS

We shall prove the following.

Theorem 3.1. Suppose that

(i) the stochastic process $(\xi_t, t \geqslant 0)$ satisfies assumptions (I) and (II) of Section 2,

(ii) the equation (1) is locally controllable,
(iii) the equation (3) is asymptotically stable.

Then there exists a neighbourhood of the origin such that the set of values of the solution of the problem (2), (4) i.e., $\{x\colon x = x_\xi(t),\, t \geqslant 0\}$ is dense in this neighbourhood with probability one.

The problem (2), (4′) where

$$(4') \qquad x(t_0) = r,\ r \in R^d$$

was considered by A. Lasota and A. Moro in [2] under assumptions of complete controllability of the system (1) for $u(\cdot)$ belonging to $U_{[-a,a]}$ $(a > 0)$ and asymptotic stability for the system (3). They proved that for almost all sample functions $\xi(t)$ the set $\{x_\xi(t),\, t \geqslant t_0\}$, where x_ξ is the solution of the initial value problem (2), (4′) with fixed r, is dense in R^d.

The theorem 3.1 can be extended to the case of a $d \times d$ matrix-valued function $A(t)$, periodic with period one and belonging to the appropriate class of differentiability.

4. PROOF OF THEOREM

We shall use two lemmas:

Lemma 4.1 If the assumption (I) is fulfilled and the equation (3) is asymptotically stable, then there exists $N > 0$ such that the solution x_ξ of the problem (2), (4) satisfies the inequality

$$\liminf_n |x_\xi(n)| \leqslant N \quad (n \text{ integer})$$

with probability equal to one.

Lemma 4.2 If assumptions (I) and (II) are fulfilled, then for every $\varepsilon > 0$, $T > 0$ and for every increasing continuous real-valued function h defined on $[0, T]$ such that $h(0) = 0$, we have

$$\operatorname{Prob}(\sup|\xi(t) - h(t)| \leqslant \varepsilon\colon\ t \in [0, T]) > 0 .$$

For proofs of these lemmas see [2]. Lemma 4.1 has even been proved for periodic matrices. Lemma 4.2 was stated in [2] for an arbitrary continuous $h(t)$ which was not necessarily increasing, with the assumption that $G(\lambda) > 0$ for $\lambda > 0$. In our case $h(t)$ is increasing and $G(\lambda) = 0$ for $\lambda \leqslant 0$. But the proof of Lemma 4.2 is similar and will be omitted.

Proof of Theorem 3.1 — Let $\Theta_0 \subset R^d$ be that neighbourhood of the origin which can be attained by the solution of the problem (1), (4) in a finite time T_0, according to the Saperstone–Yorke theorem.

Let x^* be an arbitrary point of $\mathcal{O}_0$ and V_ε^* an arbitrary neighbourhood of this point with radius ε.

We shall show that the solution of the problem (2), (4) reaches the neighbourhood V_ε^* with probability equal to one.

Denote

$$K_N = \{x\colon |x| \leqslant N\}$$

where N is chosen as in Lemma 4.1.

By the assumption of local controllability there exists an integrable function $u_0(t)$, finite time T_0 and the appropriate solution $x_0(t)$ ($t \in [0, T_0]$) of the problem (1), (4) which reaches the point x^* i.e., $x^* = x_0(T_0)$.

By using the continuous dependence property of the solution upon initial conditions it follows that there exists a positive number δ such that every solution of the equation (1), starting from the neighbourhood V_δ with the controller $u_0(\cdot)$, reaches the neighbourhood $V_{\varepsilon/2}^*$.

Furthermore, by the assumption of asymptotic stability, the solution $x(t)$ of the equation (1) starting from an arbitrary point of K_N with the controller identically equal to zero reaches the neighbourhood V_δ in finite time T_p which we can estimate from the inequality

$$|x(t)| \leqslant r|p|e^{-at}.$$

The constants a and r are positive and depend on the matrix A of the system (3).

Choose T_p in such manner that the following inequality holds true

$$r|p|e^{-aT_p} < \delta$$

i.e.,

$$T_p > \frac{1}{a}\ln\frac{r|p|}{\delta}.$$

We define the controller $u_p(\cdot)$ as follows

$$u_p(t) = \begin{cases} 0, & 0 \leqslant t \leqslant T_p \\ u_0(t-T_p), & T_p \leqslant t \leqslant T_0+T_p. \end{cases}$$

Let

$$f_p(t) = \int_0^t u_p(s)\,ds.$$

By using once more the continuous dependence property of the solution upon a parameter and initial conditions, we conclude that for every point p of K_N there exist such neighbourhood V_p and a positive number ε_p^* that the solution of the equation

$$\dot{x} = Ax + bh', \quad x(0) \in V_p$$

where

$$|h(t) - f_p(t)| \leqslant \varepsilon_p^*$$

reaches the neighbourhood V_ε^*.