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## An Application of the Homotopy Method to Linear Problems for Nonlinear Differential Systems

1. For systems of differential equations

$$(1) \quad x'(t) = f(t, x(t))$$

where

$$f(t, x) = (f_1(t, x_1, \dots, x_n), \dots, f_n(t, x_1, \dots, x_n))$$

we consider the so-called Nicoletti problem: given  $n$  distinct points  $t_1 < \dots < t_n$  in the interval  $[a, b]$  and  $n$  real numbers  $r_1, \dots, r_n$ , does there exist a solution  $x(t) = (x_1(t), \dots, x_n(t))$  of equation (1) such that

$$(2) \quad x_i(t_i) = r_i \quad (i = 1, \dots, n)?$$

The purpose of this paper is to give an existence theorem for a special Nicoletti problem for systems of differential equations (1) in the proof of which we use the homotopy method.

The formulation of our theorem is similar to the formulation of theorem in [2].

In the proof of their theorem Lasota and Opial have used the Schauder fixed-point theorem.

We shall show that this theorem may also be proved by an application of the homotopy method in a way very similar to the method used in the proof of our theorem.

Our method is similar to that of [3].

2. We shall assume that the function  $f(t, x)$  is defined and continuous in the set  $[a, b] \times R^n$ . Furthermore, we assume that there exist  $n^2$  nonnegative continuous functions  $P_{ij}(t)$  ( $i, j = 1, \dots, n$ ) and  $n$  nonnegative continuous functions  $P_i(t)$  ( $i = 1, \dots, n$ ) such that

$$(3) \quad |f_i(t, x_1, \dots, x_n)| \leq P_i(t) + \sum_{j=1}^n P_{ij}(t)|x_j| \quad \text{in } [a, b] \times R^n \text{ for } i = 1, \dots, n$$

**Theorem.** If there exists an  $\varepsilon > 0$ , such that every function  $x(t)$  satisfying the differential inequality

$$|x'_i(t)| \leq \sum_{j=1}^n (P_{ij}(t) + \varepsilon) |x_j(t)| \quad (i = 1, \dots, n)$$

and the condition

$$x_i(t_i) = 0 \quad (i = 1, \dots, n)$$

is identically equal to zero, then problem (1), (2) has solutions.

**Proof.** Consider the first case when (A) for every system of differential equations

$$(4) \quad x'(t) = \lambda f(t, x(t)), \quad \lambda \in [0, 1]$$

the Cauchy problem with arbitrary initial data has a unique solution, then it is sufficient to show that

- (i) For  $\lambda = 0$  system (4) has an odd number of solutions satisfying (2),
- (ii) If solutions to (4), (2) exist, they are bounded for every  $\lambda \in [0, 1]$  by a common constant.

Indeed, from Theorem II on the properties of a deformation by homotopy and by the considerations contained in [3], it follows that there exist solutions of problem (1), (2). It is easily seen that condition (i) is satisfied. For  $\lambda = 0$  problem (4), (2) has the unique solution  $x(t) = (r_1, \dots, r_n)$ . Therefore, it is sufficient to verify condition (ii).

For arbitrary continuous vector functions  $x(t)$  on  $[a, b]$ , define  $\|x(t)\|$  by the formula:

$$\|x(t)\| = \sum_{i=1}^n \max_{[a,b]} |x_i(t)|$$

Suppose that for a given  $\lambda \in [0, 1]$  the boundary value problem (4), (2) has a solution  $x(t, \lambda) = (x_1(t, \lambda), \dots, x_n(t, \lambda))$ . We have

$$|x'_i(t, \lambda)| = |\lambda f_i(t, x_1(t, \lambda), \dots, x_n(t, \lambda))| \leq P_i(t) + \sum_{j=1}^n P_{ij}(t) |x_j(t, \lambda)|.$$

Denoting  $x_i(t, \lambda) - r_i$  by  $y_i(t, \lambda)$  ( $i = 1, \dots, n$ ) we obtain

$$(5) \quad |y'_i(t, \lambda)| \leq N + \sum_{j=1}^n P_{ij}(t) |y_j(t, \lambda)|$$

and

$$(6) \quad y_i(t_i) = 0$$

where  $N = \max_i \left( \max_{[a,b]} (P_i(t) + \sum_{j=1}^n P_{ij}(t) |r_j|) \right)$

Two cases are possible: there exists a constant  $k > 0$  such that for every solution of (5), (6) there exists  $t_0 \in [a, b]$  such that the inequality

$$\sum_{i=1}^n |y_i(t_0, \lambda)| \leq k$$

is satisfied, or for every constant  $k$  there exists a solution  $y_{i(k)}(t, \lambda)$  satisfying (5), (6) such that

$$\sum_{i=1}^n |y_{i(k)}(t, \lambda)| > k, \quad \text{for every } t \in [a, b].$$

Suppose that the first case holds.

From (5) it follows that

$$\sum_{i=1}^n |y'_i(t, \lambda)| \leq A + B \sum_{i=1}^n |y_i(t, \lambda)|$$

where  $A = nN$  and  $B = n \cdot \max_{i,j} (\max_{[a,b]} P_{ij}(t))$ .

This implies by a well-known theorem on differential inequalities ([1], p. 151) that

$$(7) \quad \sum_{i=1}^n |y_i(t, \lambda)| \leq k \exp(B(b-a)) + \frac{A}{B} (\exp B(b-a) - 1)$$

and this means that for the given  $\lambda$  the solution of (4), (2) is bounded in the sense of norm  $\| \cdot \|$ . Now we shall show the second case is impossible.

We have

$$|y'_{ik}| \leq N + \sum_{j=1}^n P_{ij}(t) |y_{j(k)}| = \sum_{j=1}^n \left( P_{ij}(t) + \frac{N}{\sum_{i=1}^n |y_{j(k)}|} \right) |y_{j(k)}| \leq \sum_{j=1}^n \left( P_{ij}(t) + \frac{N}{k} \right) |y_{j(k)}|$$

For a sufficiently large  $k$ ,

$$|y'_{i(k)}| \leq \sum_{j=1}^n (P_{ij}(t) + \epsilon) |y_{j(k)}|.$$

From the assumption of our Theorem, this implies that  $y_{i(k)} = 0$ .

This completes the proof in the case (A).

Suppose now that (A) is not satisfied.

Let  $\eta > 0$ . Put  $\bar{P}_i(t) = P_i(t) + \eta$ ,  $\bar{N} = N + \eta$ ,  $\bar{A} = n\bar{N}$ .

There exists a constant  $k$  such that for every solution  $y(t)$  of the differential inequality

$$|y'_i(t)| \leq \bar{N} + \sum_{j=1}^n \bar{P}_{ij}(t) |y_j(t)|$$

such that

$$y_i(t_i) = 0$$

the inequality

$$(7') \quad \sum_{i=1}^n |y_i(t)| \leq \bar{k} \exp(B(b-a)) + \frac{\bar{A}}{B} (\exp B(b-a) - 1)$$

is satisfied.

Denote by  $l$  the right side of (7'), and by  $\Delta$  the interval  $[l + \sum_{i=1}^n r_i, -l - \sum_{i=1}^n r_i]$ .

There exists a sequence  $\{f_n(t, x)\}$  of functions defined in the set  $[a, b] \times \Delta^n$ , satisfying conditions

$$(3') \quad |f_{ni}(t, x_1, \dots, x_n)| < \overline{P_i(t)} + \sum_{j=1}^n P_{ij}(t) |x_j|$$

and (A) such that

$$f(t, x) = \lim_{n \rightarrow \infty} f_n(t, x)$$

uniformly on  $[a, b] \times R^n$ .

From the first part of the proof it follows that there exists a solution  $x_n(t)$  of the equations

$$x' = f_n(t, x) \quad (n = 1, 2, \dots)$$

satisfying condition (2).

By (7'), the sequence  $\{x_n(t)\}$  is uniformly bounded and by (3') it is equicontinuous. Thus, by the theorem of Arzela, there exists a uniformly convergent subsequence  $\{x_{n(k)}(t)\}$ . Its limit is a solution of the problem (1), (2).

This completes the proof.

3. We consider the Nicoletti problem for an  $n$ th-order differential equation which has already been studied in [2].

For the  $n$ th-order differential equation

$$(8) \quad x^{(n)} = f(t, x, x', \dots, x^{(n-1)}),$$

we consider the following problem: given  $n$  distinct points  $t_1 < \dots < t_n$  in the interval  $[a, b]$  and  $n$  arbitrary real numbers  $r_1, \dots, r_n$ , does there exist a solution  $x(t)$  of (8) such that

$$(9) \quad x(t_i) = r_i \quad (i = 1, \dots, n)?$$

We shall assume that the function  $f(t, x) = f(t, x_0, \dots, x_{n-1})$  is defined and continuous in the set  $[a, b] \times R^n$ .

Furthermore, we assume that there are  $n$  nonnegative continuous functions  $P_i(t)$  ( $i = 0, \dots, n-1$ ) and a nonnegative constant  $M$  such that

$$|f(t, x)| \leq M + \sum_{i=0}^{n-1} P_i(t) |x_i|.$$

