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The Degenerate Case of Boundary Value Problems for Nonlinear Differential Systems

1. INTRODUCTION

Consider the linear system of differential equations

$$x' = A(t)x + f(t) \tag{1}$$

where $A(t)$ is an $n\times n$ matrix continuous on an interval I, x and $f(t)$ are n-dimensional vectors and $f(t)$ is continuous on I. We may assume, without loss of generality, that $I\supset [0, 1]$.

Denote by $\| \ \|$ the Euclidean norm of vectors, and denote by C^n the space of all continuous n-dimensional vector functions $x(t)$ on I with the norm $\|x(t)\|_n = \sup\{\|x(t)\|:\ t\in I\}$.

For a given linear continuous mapping L of C^n into R^n and for a given point l in R^n, the general boundary value problem is usually stated as follows: does there exist a solution $x(t)$ of (1) such that

$$Lx = l\,? \tag{2}$$

It is well known that if for the fundamental matrix $\Phi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$ of the corresponding homogeneous system

$$x' = A(t)x$$

the rank of the matrix

$$L\Phi(t) = (L\varphi_1(t), \ldots, L\varphi_n(t))$$

is equal to n, then the boundary value problem (1), (2) has one and only one solution for any l. In the case where the rank of the matrix $L\Phi(t)$ is smaller than n the problem of existence of solutions of (1) satisfying (2) becomes much more delicate. This is discussed in Section 2 (see also [2]).

The same existence problem may be stated for nonlinear systems of differential equations of the form

$$(3) \qquad x' = A(t)x + f(t) + \varepsilon X(t, x, \varepsilon)$$

where ε is a parameter, $X(t, x, \varepsilon)$ is an n-dimensional vector continuous for $(t, x) \in I \times R^n$ and $|\varepsilon| < \varepsilon_0$.

A particular case of problem (3), (2), namely the N-point boundary value problem of the form

$$\sum_{i=0}^{N} M_i x(t_i) = l$$

where M_i $(i = 0, \ldots, N)$ are given constant square matrices, $t_i \in I$, $t_0 < t_1 < \ldots < t_N$, has been considered by Urabe [2].

By a method analogous to that of Urabe we obtain in Section 3 a simple necessary condition in order that the problems (3), (2) have a solution.

2. LINEAR PROBLEMS FOR LINEAR SYSTEMS IN THE DEGENERATE CASE

Suppose that the rank of the matrix $L\Phi(t)$ is $n-m$ $(1 \leqslant m \leqslant n)$. The following theorem gives a sufficient and necessary condition in order that the boundary value problem (1), (2) have at least one solution.

Theorem 0. System (1) possesses a solution satisfying the boundary condition (2) if and only if

$$(4) \qquad \Delta l - \Delta L\Big(\Phi(t) \int_0^t \Phi^{-1}(s) f(s)\, ds\Big) = 0$$

where Δ is an $m \times n$ matrix whose row vectors d_a $(a = 1, \ldots, m)$ are linearly independent, satisfying the condition

$$d_a L\Phi(t) = 0\,.$$

If (4) is valid for a given l and $f(t)$, then any solution $x(t)$ of (1), (2) can be given by the formula

$$(5) \quad x(t) = \Phi(t) \sum_{a=1}^{m} k_a c_a(t) + \Phi(t) S\Big(l - L\Phi(t) \int_0^t \Phi^{-1}(s) f(s)\, ds\Big) + \Phi(t) \int_0^t \Phi^{-1}(s) f(s)\, ds$$

where k_a $(a = 1, \ldots, m)$ are arbitrary constants, $c_a(t)$ are m linearly independent column vectors satisfying $L\Phi(t) c_a = 0$, S is an $n \times n$ matrix, independent of $f(t)$ and l, such that

$$L\Phi(t) S p = p$$

for any column vector p satisfying

$$\Delta p = 0 .$$

Proof. Any solution of (1) can be written as

$$x(t) = \Phi(t)q + \Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds \tag{6}$$

where q is a constant vector in R^n.

Solution (6) satisfies the boundary value condition (2) if and only if

$$L(\Phi(t)q) + L\Big(\Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds\Big) = l ,$$

that is if and only if

$$(L\Phi(t))q = l - L\Big(\Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds\Big) . \tag{7}$$

From the lemma in [2] concerning linear algebraic equations it follows that the constant vector q satisfying (7) exists if and only if (4) is satisfied and that q can be given by

$$q = \sum_{\alpha=1}^{m} k_\alpha c_\alpha + S\Big[l - L\Big(\Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds\Big)\Big] . \tag{8}$$

Making use of (6) and (8) we have (5) and this completes the proof.

3. THE LINEAR PROBLEM FOR NONLINEAR SYSTEMS IN THE DEGENERATE CASE

Assume that the rank of the matrix $L\Phi(t)$ is $n-m$ $(1 \leqslant m \leqslant n)$ and define mappings F, G by the formulae

$$F(l, f(t)) = \Delta l - \Delta L\Big(\Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds\Big) ,$$

$$G(k, f(t)) = \Phi(t)\sum_{\alpha=1}^{m} k_\alpha c_\alpha + \Phi(t)S\Big(l - L\Phi(t)\int_0^t \Phi^{-1}(s)f(s) + \Phi(t)\int_0^t \Phi^{-1}(s)f(s)\,ds\Big)$$

where $k = (k_1, \ldots, k_m)$ is an m-dimensional vector. By (5), $G(k, f(t))$ is the solution to (1), (2).

Denote by $F_2(l, f(t))$ the Fréchet derivative of $F(l, f(t))$ with respect to $f(t)$. For any continuous vector function $h(t)$,

$$F_2(l, f(t))h(t) = -\Delta L\Big(\Phi(t)\int_0^t \Phi^{-1}(s)h(s)\,ds\Big) . \tag{9}$$

We have the following.

Theorem 1. If problem (1), (2) possesses solutions, then a necessary condition in order that the boundary value problem (3), (2) have a solution (for ε sufficiently small) is that there exists a solution $x_0(t)$ of (1), (2) such that

$$F_2(l, f(t))\, X(t, x_0(t), 0) = 0\,,$$

that is

$$-\Delta L\left(\Phi(t)\int_0^t \Phi^{-1}(s)\, X(s, x_0(s), 0)\, ds\right) = 0\,.$$

Proof. Suppose that for any small ε, $|\varepsilon| > 0$, the boundary value problem (3), (2) has a solution $x_\varepsilon = x_\varepsilon(t)$. By Section 2, we have

(10) $$x_\varepsilon(t) = G\big(k_\varepsilon, f(t) + \varepsilon X(t, x_\varepsilon(t), \varepsilon)\big)$$

and

(11) $$F\big(l, f(t) + \varepsilon X(t, x_\varepsilon(t), \varepsilon)\big) = 0\,.$$

(11) can be written in the form

(12) $$F\big(l, f(t) + \varepsilon X(t, x_\varepsilon(t), \varepsilon)\big) = F\big(l, f(t)\big) + \varepsilon \int_0^1 F_2\big(l, f(t) + r\varepsilon X(t, x_\varepsilon(t), \varepsilon)\big)\, X(t, x_\varepsilon(t), \varepsilon)\, dr = 0\,.$$

From (9) we have

(13) $$F_2\big(l, f(t) + r\varepsilon X(t, x_\varepsilon(t), \varepsilon)\big) = F_2\big(l, f(t)\big)\,.$$

By substituting (13) into (12) we obtain

(14) $$F_2(l, f(t))\, X(t, x_\varepsilon(t), \varepsilon) = 0\,.$$

Substituting (10) into (14) we have

$$F_2(l, f(t))\, X\Big(t, G\big(k_\varepsilon, f(t) + \varepsilon X(t, x_\varepsilon(t), \varepsilon)\big), \varepsilon\Big) = 0$$

which, as $\varepsilon \to 0$, tends to the equality

(15) $$F_2(l, f(t))\, X\big(t, G(k_0, f(t)), 0\big) = 0\,.$$

Denoting $G(k_0, f(t))$ by $x_0(t)$ we have

$$F_2(l, f(t))\, X(t, x_0(t), 0) = 0$$

and this completes the proof.

Now assume that $X(t, x, \varepsilon)$ is of class C^2 with respect to x and ε. Denote by $\psi(t, x, \varepsilon)$ the Jacobian matrix of $X(t, x, \varepsilon)$ with respect to x. We have the following.

Theorem 2. If there exist $k_a = k_a^0$ ($a = 1, \ldots, m$) satisfying (15) and if the Jacobian matrix J of the left-hand side of (15) with respect to k_a ($a = 1, \ldots, m$) is non-singular for $k_a = k_a^0$ ($a = 1, \ldots, m$), then there exists $\varepsilon_1 > 0$ such that for any ε, $0 < |\varepsilon| < \varepsilon_1$ the boundary value problem (3), (2) possesses a solution.

Proof. Take the system of equations

$$(16)\qquad \begin{aligned} T_1(x,k;\ \varepsilon) &= x(t)-G\big(k,f(t)+\varepsilon X(t,x(t),\varepsilon)\big)=0\\ T_2(x,k;\ \varepsilon) &= F_2(l,f(t))X\big(t,G(k,f+\varepsilon X(t,x,\varepsilon),\varepsilon)\big)=0\,. \end{aligned}$$

It is clear that, for $\varepsilon \neq 0$, $\{x(t), k\}$ (with some k) is a solution of (16) if and only if $x(t)$ is a solution of the boundary value problem (3), (2). Thus, the problem of existence of solutions to (3), (2) reduces to the problem of existence of solutions to system (16) of two functional equations. Put

$$T(x,k;\ \varepsilon)=\{T_1(x,k;\ \varepsilon), T_2(x,k;\ \varepsilon)\}\,.$$

Then equation (16) can be written as

$$(17)\qquad T(x,k;\ \varepsilon)=0$$

and can be regarded as an equation, dependent on the parameter ε, in the Banach space $C^n\times R^n$ with norm $\|\{x(t),k\}\|_0=\|x(t)\|_n+\|k\|$. We have

$$(18)\qquad T(x_0,k_0;\ 0)=0\,.$$

Let $J_{i1}(x,k;\ \varepsilon)$, $J_{i2}(x,k;\ \varepsilon)$ $(i=1,2)$, respectively, be the Fréchet derivatives of T_i $(x,k;\ \varepsilon)$ with respect to x and k. Put

$$J(x,k;\ \varepsilon)=\begin{bmatrix} J_{11}(x,k;\ \varepsilon) & J_{12}(x,k;\ \varepsilon)\\ J_{21}(x,k;\ \varepsilon) & J_{22}(x,k;\ \varepsilon)\end{bmatrix}.$$

Then evidently $J(x,k;\ \varepsilon)$ is the Fréchet derivative of $T(x,k;\ \varepsilon)$ with respect to $\{x,k\}$.

For any continuous vector function $h(t)$ and any m-dimensional vector $\lambda=(\lambda_1,\ldots,\lambda_m)$

$$J_{11}h(t)=h(t)+\varepsilon\Big[\Phi(t)SL\Phi(t)\int_0^t\Phi^{-1}(s)\Psi(s,x(s),\varepsilon)h(s)ds-\\ -\Phi(t)\int_0^t\Phi^{-1}(s)\Psi(s,x(s),\varepsilon)h(s)ds\Big]$$

$$J_{12}\lambda=-\Phi\sum_{i=1}^m\lambda_a c_a$$

$$J_{21}h(t)=\varepsilon\Big\{-\Delta L\Phi(t)\int_0^t\Big[\Phi^{-1}(s)\Psi\big(s,G(k,f+\varepsilon X(s,x(s),\varepsilon)),\varepsilon\big)\times\\ \times\Big(-\Phi(s)SL\Phi(s)\int_0^s\Phi^{-1}(\sigma)\Psi(\sigma,x(\sigma),\varepsilon)h(\sigma)d\sigma+\\ +\Phi(s)\int_0^s\Phi^{-1}(\sigma)\Psi(\sigma,x(\sigma),\varepsilon)h(\sigma)d\sigma\Big)\Big]ds\Big\}$$

$$j_{22}\lambda=\Delta L\Phi(t)\int_0^t\Phi^{-1}(s)\Big[\Psi\big(s,G(k,f+\varepsilon X(s,x(s),\varepsilon)),\varepsilon\big)\cdot\sum_{a=1}^m\lambda_a c_a ds\Big]\,.$$