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Some Remarks on Processes

1. We recall the following definitions (see [1], [2]).

Definition 1. A process on X (X is a metric space with metric d) is a mapping

$$u\colon R\times X\times R_* \to X$$

with the following properties

(i) $u^t(x, 0) = x$ for all $t \in R$, $x \in X$,

(ii) $u^t(x, s+\tau) = u^{t+s}(u^t(x, s), \tau)$ for all $t \in R$, $x \in X$, $\tau, s \in R_*$,

(iii) for fixed $\tau \in R_*$, the one parameter family of maps

$$u^t(\cdot, \tau)\colon X \to X, \text{ parameter } t \in R,$$

is equicontinuous.

Definition 2. The orbit of u which originates at (t, x) is the map

$$u^t(x, \cdot)\colon R_* \to X.$$

Definition 3. A process u on X is called continuous if for any fixed $x \in X$, $\tau \in R_*$, the map

$$u^{\cdot}(x, \tau)\colon R \to X$$

is continuous on R.

Remark 1. Remark 2.1 in paper [1] said that every process on X can be extended on to a process on the completion of X. This is false without additional assumptions.

Example 1. $X = R\times (R\setminus\{0\})$, d is the restriction of the Euclidean metric on the plane,

$$u^t(x, \tau) = (x_1 + \tau\operatorname{sgn} x_2, x_2), \quad x = (x_1, x_2) \in R\times (R\setminus\{0\}) \quad t \in R,\ \tau \in R_*,$$

Proposition 1. If for a fixed $\tau \in R_*$, the one parameter family of maps

$$u^t(\cdot, \tau)\colon X \to X, \text{ parameter } t \in R,$$

is uniformly equicontinuous, then the process u on X can be extended synonymously on to a process on the completion of X.

Proof. Let $\widetilde{X}$ be a completion of X. For $x \in \widetilde{X}$, $t \in R$, $\tau \in R_*$ we put

$$\widetilde{u}^t(x, \tau) = \lim_{n\to\infty} u^t(x, \tau), \quad \text{where } x_n \to x,\ x_n \in X.$$

Definition $\widetilde{u}^t(x, \tau)$ is not dependent on sequence $\{x_n\}$, the mapping $\widetilde{u}$ satisfies the conditions (i), (ii). We show (iii).

Let $\tau \in R_*$, $x \in \widetilde{X}$, $\varepsilon > 0$ be fixed. With the uniform equicontinuity there exists $\delta > 0$ such that $d(u^t(z_1, \tau), u^t(z_2, \tau)) < \varepsilon$ for every $t \in R$, $z_1, z_2 \in X$, $d(z_1, z_2) < \delta$.

We take $y \in \widetilde{X}$, $d(y, x) < \delta$. Let $x_n \to x$, $y_n \to y$, $x_n, y_n \in X$. Using the triangle inequality

$$d(x_n, y_n) \leqslant d(x_n, x) + d(x, y) + d(y_n, y)$$

it is deduce that $d(x_n, y_n) < \delta, n \geqslant N$ for some N. Thus

$$d(u^t(x_n, \tau), u^t(y_n, \tau)) < \varepsilon.$$

Hence

$$d(u^t(y, \tau), u^t(x, \tau)) = d(\lim_{n\to\infty} u^t(y_n, \tau), \lim_{n\to\infty} u^t(x_n, \tau)) = \lim_{n\to\infty} d(u^t(y_n, \tau), u^t(x_n, \tau)) \leqslant \varepsilon.$$

The condition (iii) is satisfied, the reste is trivial.

Remark 2. If the orbits of a process u are continuous, then u can be not continuous. There are processes u which are not continuous but have continuous orbits (compare Remark 2.2 in [1]).

Example 2. $X = R$, d is the Euclidean metric,

$$u^t(x, \tau) = \begin{cases} x & \text{for} \quad t \geqslant 0 \\ x\dfrac{t+\tau}{t} & \text{for} \quad t < 0,\ t+\tau < 0 \\ 0 & \text{for} \quad t < 0,\ t+\tau \geqslant 0. \end{cases}$$

The inequality (2.3) in Remark 2.2 in [1] implies only the right-hand continuity.

2. We recall a pseudoprocess definition (see [3]). Let X be an abstract space, let $(W, +)$ be a group, let $U \subset W$ be a subsemigroup, $0 \in U$. Let u: $W \times X \times U \to X$.

Definition 4. A system (X, W, U, u) is said to be a pseudoprocess on X if:

(a) $u^t(x, 0) = x$ for all $t \in W$, $x \in X$,

(b) $u^t(x, s+\tau) = u^{t+s}(u^t(x, s), \tau)$ for all $t \in W$, $x \in X$, $\tau, s \in U$.

Definition 5. A pseudoprocess (X, W, U, u) is called periodic with period $T \in U$ if $u^{t+T}(x, \tau) = u^t(x, \tau)$ for each $t \in W$, $x \in X$, $\tau \in U$.

We assume that $W = R$, $U = R_*$, $X = R^n$. We shall give some characterization of a pseudoprocess defined by an ordinary differential equation.

Let f: $R \times R^n \to R^n$ be continuous. We consider the differential equation:

$$x' = f(t, x) \tag{1}$$

We assume that for every $(t_0, x_0) \in R\times R^n$ there exists a sole solution $\varphi_{(t_0,x_0)}: [t_0, \infty) \to R^n$ satisfying $x(t_0) = x_0$ and equation (1). We put:

$$u^{t_0}(x_0, \tau) = \varphi_{(t_0,x_0)}(t_0+\tau)$$

(R^n, R, R_*, u) is a pseudoprocess.

Theorem. Let f satisfies the condition:

$$|f(t, x)-f(t, y)| \leqslant \omega(|x-y|)$$

where ω is an increasing function and 0 is the right-hand maximum solution of problem $x'(t) = \omega(x(t))$, $x(0) = 0$. Let $T > 0$. Then: $f(\cdot, x)$ is a periodic with period T for every $x \in R^n \Leftrightarrow$ the pseudoprocess (R^n, R, R_*, u) is periodic with period T.

Proof

$\Leftarrow$ We have

$$u^0(x_0, \tau) = u^T(x_0, \tau) \text{ for every } x_0 \in R^n,\ \tau \in R_* .$$

Hence

$$\varphi_{(0,x_0)}(\tau) = \varphi_{(T,x_0)}(T+\tau)$$

$$\frac{d}{d\tau}\varphi_{(0,x_0)}(\tau) = \frac{d}{d\tau}\varphi_{(T,x_0)}(T+\tau)$$

$$f(\tau, x_0) = f(T+\tau, x_0)$$

$\Rightarrow$ Let $x_0 \in R^n$, $t_0 \in R$ be fixed. We show that

$$u^{t_0+T}(x_0, \tau) = u^{t_0}(x_0, \tau) \text{ for every } \tau \in R_* .$$

$$\begin{aligned}
|u^{t_0+T}(x_0, \tau)- u^{t_0}(x_0, \tau)| &= |\varphi_{(t_0+T,x_0)}(t_0+T+\tau)-\varphi_{(t_0,x_0)}(t_0+\tau)| \\
&= \left|\left[x_0+\int_{t_0+T}^{t_0+T+\tau} f(s, \varphi_{(t_0+T,x_0)}(s))\, ds\right]-\left[x_0+\int_{t_0}^{t_0+\tau} f(s, \varphi_{(t_0,x_0)}(s))\, ds\right]\right| \\
&= \left|\int_{t_0}^{t_0+\tau} [f(s, \varphi_{(t_0+T,x_0)}(s+T))-f(s, \varphi_{(t_0,x_0)}(s))]\, ds\right| \\
&\leqslant \int_{t_0}^{t_0+\tau} \omega(|\varphi_{(t_0+T,x_0)}(s+T)-\varphi_{(t_0,x_0)}(s)|)\, ds \\
&= \int_0^{\tau} \omega(|u^{t_0+T}(x_0, s)-u^{t_0}(x_0, s)|)\, ds
\end{aligned}$$

We put $h(\tau) = |u^{t_0+T}(x_0, \tau)- u^{t_0}(x_0, \tau)|$ and we have an integral inequality

$$h(\tau) \leqslant \int_0^{\tau} \omega(h(s))\, ds \quad \text{for } \tau \in R_* .$$

In virtue of Theorem 22.1 (see [4]) we have

$$h(\tau) = 0 \quad \text{for each } \tau \in R_* .$$

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