

Lukasz Kamykowski

### On a Certain Generalization of Schreier's Theorem in Lattices

Let  $X$  be a set and  $\bar{R}$  a binary relation  $\bar{R} \subseteq X \times X$  such that:

- (1)  $\forall a \in X: a \bar{R} a$  (reflexive)
- (2)  $\forall a, b \in X: a \bar{R} b, b \bar{R} a \Rightarrow a = b$  (antisymmetric)
- (3)  $\forall n \in \mathbb{N}, \forall x_1, \dots, x_n \in X: (x_1 \bar{R} x_2, x_2 \bar{R} x_3, \dots, x_{n-1} \bar{R} x_n; x_1 \bar{R} x_n) \Rightarrow x_1 \bar{R} x_i$   
( $i = 1, \dots, n$ )

Condition (3) is weaker than a transitive condition. For a transitive relation  $\leq$  we should have  $x_i \leq x_j$  if only  $i \leq j$  as the consequent of the implication.

Let now  $A$  be a group and  $\mathcal{G}$  a set of all subgroups of  $A$ .

We define a relation  $\bar{R}$  on  $\mathcal{G}$  as follows:

$\forall H, K \in \mathcal{G}: K \bar{R} H \stackrel{\text{df}}{\Leftrightarrow} K \subseteq H$  and  $\forall h \in H: hK = Kh$ ; that is to say  $K$  is an invariant subgroup of  $H$ . It is easy to prove that  $(\mathcal{G}, \bar{R})$  satisfies (1)–(3). We shall demonstrate here only (3). Let  $H_1 \bar{R} H_2, \dots, H_{n-1} \bar{R} H_n$  and  $H_1 \bar{R} H_n$ . Let also  $i \in \{1, \dots, n\}$ . We get  $H_1 \subseteq H_i, H_i \subseteq H_n$  and  $\forall h \in H_n, hH_1 = H_1h$  therefore  $\forall h' \in H_i, h'H_1 = H_1h'$  so that  $H_1 \bar{R} H_i$ .

Return now to the general situation. Let

$$(x_1, \dots, x_n) \in X^n (n \geq 2) (x: \{1, \dots, n\} \rightarrow X).$$

The sequence  $x = (x_1, \dots, x_n)$  is called a chain from  $a$  to  $b$  if and only if:

- 1)  $x_1 = a, x_n = b$ ,
- 2)  $x_1 \bar{R} x_2, x_2 \bar{R} x_3, \dots, x_{n-1} \bar{R} x_n$  (we shall note it more shortly  $x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n$  or in diagrams:  $x_1 \Rightarrow x_2 \Rightarrow x_3 \Rightarrow \dots \Rightarrow x_n$ ). The chain  $x$  will be called a proper chain if  $x$  is a one-to-one mapping. For a proper chain  $(x_1, \dots, x_n)$  we have  $x_i \bar{R} x_{i+1}$  and  $x_i \neq x_{i+1}$  ( $i = 1, \dots, n$ ). We shall note it:  $x_i R x_{i+1}$  and in diagrams:  $x_i \rightarrow x_{i+1}$ . The following property of relations which satisfy (1)–(3) can be proved:

- (I) Let  $(x_1, \dots, x_n)$  be a chain from  $a$  to  $a$  ( $a = x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n = a$ ). Then  $a = x_1 = x_2 = \dots = x_{n-1} = x_n$ . So that the situation represented by Fig. 1. is impossible.

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- (3)  $\forall n \in \mathbb{N}, \forall x_1, \dots, x_n \in X: (x_1 \bar{R} x_2, x_2 \bar{R} x_3, \dots, x_{n-1} \bar{R} x_n; x_1 \bar{R} x_n) \Rightarrow x_1 \bar{R} x_i$   
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Return now to the general situation. Let

$$(x_1, \dots, x_n) \in X^n (n \geq 2) (x: \{1, \dots, n\} \rightarrow X).$$

The sequence  $x = (x_1, \dots, x_n)$  is called a chain from  $a$  to  $b$  if and only if:

- 1)  $x_1 = a, x_n = b$ ,
- 2)  $x_1 \bar{R} x_2, x_2 \bar{R} x_3, \dots, x_{n-1} \bar{R} x_n$  (we shall note it more shortly  $x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n$  or in diagrams:  $x_1 \Rightarrow x_2 \Rightarrow x_3 \Rightarrow \dots \Rightarrow x_n$ ). The chain  $x$  will be called a proper chain if  $x$  is a one-to-one mapping. For a proper chain  $(x_1, \dots, x_n)$  we have  $x_i \bar{R} x_{i+1}$  and  $x_i \neq x_{i+1}$  ( $i = 1, \dots, n$ ). We shall note it:  $x_i R x_{i+1}$  and in diagrams:  $x_i \rightarrow x_{i+1}$ . The following property of relations which satisfy (1)–(3) can be proved:

- (I) Let  $(x_1, \dots, x_n)$  be a chain from  $a$  to  $a$  ( $a = x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n = a$ ). Then  $a = x_1 = x_2 = \dots = x_{n-1} = x_n$ . So that the situation represented by Fig. 1. is impossible.

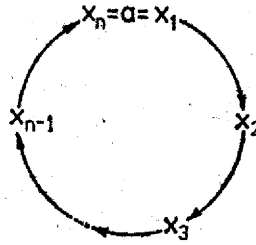


Fig. 1

We prove the lemma by induction on  $n$ . For  $n = 2$   $x_1 = a = x_2$ . Suppose that the lemma is valid for some  $n$  and  $x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n \bar{R} x_{n+1} = x_1$ . Since  $\bar{R}$  is reflexive  $x_1 \bar{R} x_{n+1}$  so that  $x_1 \bar{R} x_n$  because of (3) but also  $x_n \bar{R} x_1 (= x_{n+1})$ . Hence by (2)  $x_1 = x_n$  and  $x_1 = x_2 = \dots = x_n = x_{n+1}$ .

(II) As a corollary of lemma (I) we obtain that the relation  $\bar{R}$  can be extended to a partial order.

Let by  $a \leq b$  ( $a - = \rightarrow b$ ) be meant that there exists some chain  $(x_1, \dots, x_n)$  from  $a$  to  $b$ . Clearly such a defined relation is transitive (if  $(x_1, \dots, x_n)$  is a chain from  $a$  to  $b$  and  $(y_1, \dots, y_m)$  is a chain from  $b$  to  $c$  then  $(x_1, \dots, x_n = y_1, \dots, y_m)$  is a chain from  $a$  to  $c$ ) and reflexive as an extension of reflexive  $\bar{R}$ . Let  $a \leq b$  and  $b \leq a$  and let  $(x_1, \dots, x_n)$  be a chain from  $a$  to  $b$ ,  $(y_1, \dots, y_m)$  a chain from  $b$  to  $a$ . Then  $(x_1, \dots, x_n = y_1, \dots, y_m)$  is a chain from  $a$  to  $a$  and according to (I)  $x_1 = y_1$ ,  $a = b$ . Hence  $\leq$  is also antisymmetric. (3) implies one more property of  $\leq$ : if  $a \bar{R} b$  and  $a \leq c \leq b$  then  $a \bar{R} c$ . The relation defined above will be called an order induced by  $\bar{R}$ .

We assume two other properties on  $\bar{R}$ :

(4) (a meet for covered pairs of elements — Fig. 2)

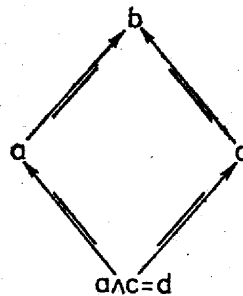


Fig. 2

$\forall a, b, c \in X$  if  $a \bar{R} b$ ,  $c \bar{R} b$  then  $\exists d \in X$ :  $d \bar{R} a$ ,  $d \bar{R} b$ ,  $d \bar{R} c$  and  $d' \bar{R} a$ ,  $d' \bar{R} c$  implies  $d' \leq d$ .

Since  $\leq$  is antisymmetric it is obvious that  $d$  is uniquely determined. We now note it by  $a \wedge_b c$ .  $a \wedge_b c$  does not depend on concrete  $b$ . Let  $b$  and  $b'$  be two elements

such that  $a \bar{R} b, c \bar{R} b, a \bar{R} b', c \bar{R} b'$ . Then we have  $a \wedge c$  and  $a \wedge c$  but  $a \wedge c \bar{R} a, a \wedge c \bar{R} c$  so  $a \wedge c \leq a \wedge c$  and also  $a \wedge c \leq a \wedge c$ . Hence  $a \wedge c = a \wedge c$  and so we shall simply write  $a \wedge c$ .

(III) If  $a, c$  in (4) satisfy  $a \leq c$  then  $a \wedge c = a$ .

Really, then there exists a chain  $a = x_1 \bar{R} x_2 \bar{R} \dots \bar{R} x_n = c$  and  $a \bar{R} b$  so (3) implies that  $a \bar{R} c$ . Obviously  $a \bar{R} a$ . Now if  $d' \bar{R} a, d' \bar{R} c$  then  $d' \leq a$  and that ends the proof.

In the case when  $X = \mathcal{G}, \bar{R} = \bar{\mathcal{R}}, H \cap K$  can be put as  $H \wedge K$  for  $H, K \in \mathcal{G}$ . If for some  $L \in \mathcal{G} H \bar{\mathcal{R}} L$  and  $K \bar{\mathcal{R}} L$  then also  $H \cap K \bar{\mathcal{R}} L$  (because for each  $l \in L$   $l(H \cap K)l^{-1} \subset H \cap K$  and this is enough for  $\forall l \in L: l(H \cap K) = (H \cap K)l$ ). As  $H \bar{\mathcal{R}} L$  and  $K \bar{\mathcal{R}} L$  it is clear that  $H \cap K \bar{\mathcal{R}} K$  and  $H \cap K \bar{\mathcal{R}} H$ . Directly can be checked that if  $M \bar{\mathcal{R}} H, M \bar{\mathcal{R}} K$  then  $M \bar{\mathcal{R}} H \cap K$ .

Note also that in any lattice  $(L, \leq)$  condition (4) is satisfied by  $\leq$  and  $d$  equal to the greatest lower bound of  $a$  and  $c$ .

Condition (5) that we assume on  $\bar{R}$  replaces partially the condition for modular lattices (Fig. 3) :

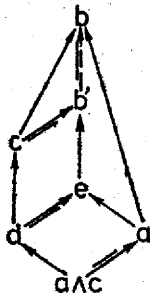


Fig. 3

$$(5) \left. \begin{array}{l} a, b, c, d \in X: a \bar{R} b \\ a \wedge c \bar{R} d \bar{R} c \bar{R} b \end{array} \right\} \Rightarrow \exists b' \leq b, \exists e: a \bar{R} e \bar{R} b' \leq b, d \bar{R} e, c \bar{R} b'$$

Consider the case  $(g, \bar{\mathcal{R}})$ . Let  $H \cap K \bar{\mathcal{R}} M \bar{\mathcal{R}} K \bar{\mathcal{R}} G, H \bar{\mathcal{R}} G$ . Then we can put  $KH$  as  $b'$  and  $MH$  as  $e$  (Fig. 4).

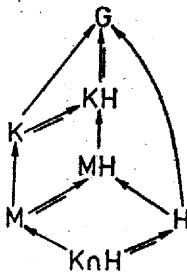


Fig. 4

