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## Analytic Continuation of Series of Homogeneous Polynomials of $n$ Complex Variables

### INTRODUCTION

Let  $f$  be an analytic function of  $n$  complex variables defined by a series of homogeneous polynomials

$$(1) \quad f(z) = \sum_{\nu=1}^{\infty} f_{\nu}(z) \quad (\deg f_{\nu} = \nu)$$

that converges in a neighbourhood of zero in the space  $C^n$ . Let  $G_f$  denote the Mittag-Leffler star of the function  $f$ , i.e. the largest starlike (with respect to zero) open set such that  $f$  can be analytically continued to  $G_f$ . The question arises how the analytic continuation of the function  $f$  to  $G_f$  can be found. If  $n = 1$  an answer is given by a known result (see e.g. [6], p. 493), which can be extended to the  $n$ -dimensional case as follows:

**Theorem 1.** There is a triangular array of complex numbers  $c_0^{\nu}, c_1^{\nu}, \dots, c_{k\nu}^{\nu}$  ( $\nu = 0, 1, \dots$ ) such that for every analytic function  $f$  given by (1) we have

$$f(z) = \sum_{\nu=0}^{\infty} [c_0^{\nu} f_0(z) + c_1^{\nu} f_1(z) + \dots + c_{k\nu}^{\nu} f_{k\nu}(z)]$$

for  $z \in G_f$ , the convergence being uniform on compact subsets of  $G_f$ .

**Proof** (due to J. Siciak). Choose a point  $z \in C^n \setminus \{0\}$  and write  $G^z = \{\lambda \in C : \lambda z \in G_f\}$ . Note that for each  $z \in G_f$ ,  $1 \in G^z$  and the function  $f_z : \lambda \rightarrow f(\lambda z)$  is analytic in  $G^z$ . So, by the Cauchy formula:

$$f(z) = (2\pi i)^{-1} \int_{\lambda_z} f(\lambda z) (\lambda - 1)^{-1} d\lambda = (2\pi i)^{-1} \int_{\lambda_z} f(\lambda z) \left(1 - \frac{1}{\lambda}\right)^{-1} \lambda^{-1} d\lambda, \quad z \in G_f,$$

where  $\gamma_z$  is any positively oriented regular Jordan curve contained in  $G^z$  such that  $1 \in \text{int} \gamma_z$  (where  $\text{int} \gamma_z$  denotes the bounded component of  $C \setminus \gamma_z$ ).

Take an arbitrary point  $z_0 \in G_f \setminus \{0\}$  and a starlike (with respect to zero) positively oriented regular Jordan curve  $\gamma$  contained in  $G^{z_0}$  so that the points  $0, 1 \in \text{int} \gamma$ . Then  $\overline{\text{int} \gamma} \cdot z_0 = \{\lambda z_0 : \lambda \in \overline{\text{int} \gamma}\}$  is contained in  $G_f$ . Since  $\overline{\text{int} \gamma}$  is compact and  $G_f$  is open one can find a number  $\varrho > 0$  such that

$$\Delta = \{\zeta \in G^m : \zeta = \lambda z, \lambda \in \overline{\text{int} \gamma}, \|z - z_0\| \leq \varrho\} \subset G_f.$$

For any  $z \in B(z_0, \varrho)$  we have

$$f(z) = (2\pi i)^{-1} \int_{\gamma} \left(1 - \frac{1}{\lambda}\right)^{-1} \frac{f(\lambda z)}{\lambda} d\lambda.$$

Note that  $\{\lambda^{-1} : \lambda \in \gamma\} \subset C \setminus [1, \infty)$ . By the Runge approximation theorem, there exists a sequence  $\{P_n\}_{n=0}^{\infty}$  of polynomials of one complex variable such that

$$(2) \quad \frac{1}{1-\tau} = \sum_{n=0}^{\infty} P_n(\tau), \quad \tau \in C \setminus [1, \infty),$$

the convergence being uniform on every compact subset of  $C \setminus [1, \infty)$ . Set

$$P_n(\tau) = c_0^n + c_1^n \tau + \dots + c_{k_n}^n \tau^{k_n}.$$

Using (2) we obtain

$$(3) \quad f(z) = (2\pi i)^{-1} \int_{\gamma} \sum_{n=0}^{\infty} P_n\left(\frac{1}{\lambda}\right) \frac{f(\lambda z)}{\lambda} d\lambda, \quad z \in B(z_0, \varrho).$$

Since series (2) converges uniformly on  $\{\lambda^{-1} : \lambda \in \gamma\}$  and since the function  $(\lambda, z) \rightarrow \frac{f(\lambda z)}{\lambda}$  is analytic in a neighborhood of  $\gamma \times \overline{B(z_0, \varrho)}$ , the right-hand side of (3) may be integrated term by term. Hence

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} \left[ c_0^n (2\pi i)^{-1} \int_{\gamma} \frac{f(\lambda z)}{\lambda} d\lambda + c_1^n (2\pi i)^{-1} \int_{\gamma} \frac{f(\lambda z)}{\lambda^2} d\lambda + \dots + c_{k_n}^n (2\pi i)^{-1} \int_{\gamma} \frac{f(\lambda z)}{\lambda^{k_n+1}} d\lambda \right] \\ &= \sum_{n=0}^{\infty} [c_0^n f_0(z) + c_1^n f_1(z) + \dots + c_{k_n}^n f_{k_n}(z)] \end{aligned}$$

for  $z \in B(z_0, \varrho)$  and the convergence is uniform in  $B(z_0, \varrho)$ . The proof is concluded.

Repeating the argument of the proof of Theorem 1 we obtain

Theorem 2 (for  $n = 1$  see [5], th. 135). Let

$$\Phi_{\delta}(\lambda) = \sum_{n=0}^{\infty} a_n(\delta) \lambda^n, \quad \lambda \in C, \delta > 0.$$

