ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCIII PRACE MATEMATYCZNE, ZESZYT 17 1975

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Remark on the Chaplygin Method for Parabolic Equations in Unbounded Domains

In paper [1] S. Brzychczy proved the existence and uniqueness in an unbounded domain D of a solution to the first Fourier problem for an almost linear, diagonal system of parabolic equations

$$\frac{\partial u_i}{\partial t} - \sum_{p=1}^{m} \frac{\partial^2 u_i}{\partial x_p^2} = f_i(t, x, u_1, \ldots, u_r) \qquad (i = 1, \ldots, r)\,, \tag{1}$$

where $(t, x) = (t, x_1, \ldots, x_m) \in D$.

Among other assumptions the right-hand sides of (1) were supposed to satisfy the following

Assumptions H

a) $f_i(t, x, u_1, \ldots, u_r)$ are continuous in $D \times R^r$ $(i = 1, \ldots, r)$ and hölderian with respect to x,

b) $f_i(t, x, u_1, \ldots, u_r)$ fulfil the Lipschitz condition with respect to $u_1, \ldots, u_r$

$$|f_i(t, x, u_1, \ldots, u_r) - f_i(t, x, \tilde{u}_1, \ldots, \tilde{u}_r)| \leqslant (L_1\|x\|^2 + L_2) \sum_{j=1}^{r} |u_j - \tilde{u}_j|\,,$$

where $\|x\|$ denotes the Euclidean norm.

Constructing a solution to the first Fourier problem for system (1) in D by means of Chaplygin's method, S. Brzychczy defined a sequence $\{u_i^{(n)}(t, x)\}$ $(i = 1, \ldots, r;\ n = 1, 2, \ldots)$ of functions, where $u_i^{(n)}(t, x)$ $(i = 1, \ldots, r)$ is a solution of class C^2 in D to the linear system

$$\frac{\partial u_i^{(n)}}{\partial t} - \sum_{p=1}^{m} \frac{\partial^2 u_i^{(n)}}{\partial x_p^2} = f_i\big(t, x, u_1^{(n-1)}(t, x), \ldots, u_r^{(n-1)}(t, x)\big) + \\ + (L_1\|x\|^2 + L_2)[u_i^{(n-1)}(t, x) - u_i^{(n)}] \qquad (i = 1, \ldots, r) \tag{2}$$

and proved its almost uniform convergence in D. Putting

$$\lim_{n\to\infty} u_i^{(n)}(t, x) = U_i(t, x)\,, \tag{3}$$

he obtained continuous functions $U_i(t, x)$ in D, which were then proved to form a solution of class C^2 in D to the system (1).

The purpose of the present paper is to prove the last fact by a shorter argument than that used by S. Brzychczy. Our argument is based on the continuous dependence of the solution in a bounded domain on the right-hand sides of equations and on the initial and boundary conditions.

We shall now prove the following theorem which implies the above statement concerning the functions $U_i(t, x)$.

Theorem. Let the functions $f_i(t, x, u_1, \ldots, u_r)$ $(i = 1, \ldots, r)$ satisfy Assumptions H. Suppose $\{u_i^{(n)}(t, x)\}$ $(i = 1, \ldots, r;\ n = 1, 2, \ldots)$ to be a sequence of functions such that $u_i^1(t, x)$ $(i = 1, \ldots, r)$ are of class C^2 in D and $u_i^{(n)}(t, x)$ $(n = 2, 3, \ldots)$ is a solution of class C^2 in D to the linear system (2). Suppose that we have (3) almost uniformly in D.

Under these assumptions $U_i(t, x)$ $(i = 1, \ldots, r)$ is a solution of class C^2 in D to the system (1).

Proof. Obviously it is sufficient to prove that in a neighbourhood of any point $(t^*, x^*) \in D$ the functions $U_i(t, x)$ $(i = 1, \ldots, r)$ form a solution of class C^2 to the system (1).

To this effect, consider a cylindrical neighbourhood of the point (t^*, x^*)

$$C = \{(t, x) : |t - t^*| < h^*,\ \|x - x^*\| < R^*\}$$

such that $\bar{C} \subset D$.

Denote

$$\Sigma = \{(t, x) : |t - t^*| < h^*,\ \|x - x^*\| = R^*\},$$

$$S_0 = \{(t, x) : t = t^* - h^*,\ \|x - x^*\| \leqslant R^*\},$$

and consider in C the linear system of equations

$$\frac{\partial z_i}{\partial t} - \sum_{p=1}^{m} \frac{\partial^2 z_i}{\partial x_p^2} = f_i\big(t, x, U_1(t, x), \ldots, U_r(t, x)\big) \quad (i = 1, \ldots, r) \tag{4}$$

with initial and boundary conditions

$$z_i(t, x) = U_i(t, x) \quad \text{for} \quad (t, x) \in S_0 \cup \Sigma . \tag{5}$$

Now, let us remark that the functions $u_i^{(n)}(t, x)$ $(i = 1, \ldots, r)$ satisfy the following system

$$\frac{\partial u_i^{(n)}}{\partial t} - \sum_{p=1}^{m} \frac{\partial^2 u_i^{(n)}}{\partial x_p^2} = f_i\big(t, x, u_1^{(n-1)}(t, x), \ldots, u_r^{(n-1)}(t, x)\big) + \\ + (L_1\|x\|^2 + L_2)[u_i^{(n-1)}(t, x) - u_i^{(n)}(t, x)] \tag{6}$$

and that by (3) they are uniformly (with respect to n) bounded in $\bar{C}$. Hence, it follows by Assumption a) that the right-hand sides of (6), which we denote shortly by $g_i^{(n)}(t, x)$ $(i = 1, \ldots, r)$, are uniformly (with respect to n) bounded in $\bar{C}$.

According to known formulas we get from (6)

$$u_i^{(n)}(t, x) = \frac{1}{(2\sqrt{\pi})^m}\left[\int\limits_{\Sigma_t} \frac{dG(t, x, \tau, \xi)}{d\nu} u_i^{(n)}(\tau, \xi)\, d\sigma + \right. \tag{7}$$

$$\left. + \int\limits_{S_0} G(t, x, t^* - h^*, \xi)\, u_i^{(n)}(t^* - h^*, \xi)\, d\xi + \int\limits_{C_t} G(t, x, \tau, \xi)\, g_i^{(n)}(\tau, \xi)\, d\tau d\xi \right],$$

where $G(t, x, \tau, \xi)$ is Green's function and Σ_t, C_t denote respectively the intersections of Σ and C with the zone $t^* - h^* < \tau < t$. By the uniform boundedness of $u_i^{(n)}$ and $g_i^{(n)}$ it follows from (7) that the functions $u_i^{(n)}(t, x)$ are continuous in $\bar{C}$ and locally Lipschitz continuous with respect to x, with a constant independent of n. Hence, we conclude by (3) that $U_i(t, x)$ are continuous in $\bar{C}$ and locally Lipschitz continuous with respect to x. The last property together with the Assumptions H implies that the right-hand sides of the system (4) are continuous in $\bar{C}$ and locally Hölder continuous with respect to x. Hence, it follows by a known theorem [2] that there exists a solution $z_i(t, x)$ to the problem (4), (5), which is of class C^2 in C.

Now, by (3) we conclude that the right-hand sides of (6) converge uniformly in $\bar{C}$ to the right-hand sides of (4) and the initial and boundary values of $u_i^{(n)}(t, x)$ converge uniformly on $S_0 \cup \Sigma$ to the respective values of $z_i(t, x)$. Hence, by a known theorem [3] (Theorem 51.1) it follows that

$$\lim_{n \to \infty} u_i^{(n)}(t, x) = z_i(t, x) \qquad (i = 1, \ldots, r) \tag{8}$$

uniformly in $\bar{C}$. The relations (3) and (8) imply that

$$U_i(t, x) = z_i(t, x) \text{ in } C$$

and consequently $z_i(t, x)$ being a solution to the system (4), of class C^2 in C, we get $U_i(t, x)$ as a solution of class C^2 in C to the system (1). This completes the proof.

I shoud like to thank Professor J. Szarski for his valuable remarks.

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