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Boundary Value Problems for Linear Differential Equations with Distributional Perturbations

In the present note we consider the problem of the existence and the uniqueness of solutions for the differential equation

(1) $$x' = Ax+f$$

satisfying the "smooth" condition

$$\langle x, \varphi\rangle = r, \text{ or more general } Lx = r\,,$$

where A is a $n\times n$ matrix with C^∞ coefficients, f is a distribution, $\varphi \in C$ has compact support and $L:(D')^n\to R^n$ is linear. Next, we derive the corollary for the nth-order scalar differential equation

$$x^{(n)}+a_1x^{(n-1)}+\ldots+a_nx = h\,,$$

where $a_i \in C^\infty$ and h is a distribution, and we give two examples of its applications. In the last section we study the problem of the existence and the uniqueness of periodic solutions for the equation (1) and the connection between the existence of periodic and bounded solutions of (1). Some results of this type are due to [2], [3].

NOTATIONS

Let R be the real line and R^n its nth-order Cartesian product. By D we denote the linear space of all infinitely differentiable functions $\varphi: R\to R$ with compact support. We shall write shortly $\Delta\varphi = \operatorname{supp}\varphi$. Let D_+ be the subset of D such that

$$D_+ = \Big\{\varphi : \varphi \in D,\, \varphi(t) \geqslant 0,\, \int_{\Delta\varphi} \varphi(s)ds = 1\Big\}\,.$$

As usual D' and $(D')^n$ will denote, respectively, the linear space of all distributions and the linear space of all n-dimensional vector distributions. For given $x \in D'$, $(x \in (D')^n)$ and $\varphi \in D$ we have $\langle x, \varphi\rangle \in R$, $(\langle x, \varphi\rangle \in R^n)$. Let C^∞ and $(C^\infty)^{n\times n}$ be, respectively, the linear space of all functions and the linear space of all $n\times n$ matrix functions defined and infinitely differentiable on R.

For a given linear mapping $L : (D')^n \to R^n$, L^n will be the induced mapping of $(D')^{n\times n}$ into $R^{n\times n}$ which to every matrix V in $(D')^{n\times n}$ assigns the matrix obtained by the application of L to every column of V. It is clear that

$$L[Vc] = [L^n V]c \tag{2}$$

for any $V \in (D')^{n\times n}$ and any $c \in R^n$.

BOUNDARY VALUE PROBLEMS

Consider a linear nonhomogeneous system of differential equations

$$x' = Ax + f \tag{3}$$

with a matrix function $A \in (C^\infty)^{n\times n}$ and a vector distribution $f \in (D')^n$. Moreover, consider a boundary value condition

$$Lx = r\,, \tag{4}$$

where L is linear mapping, $L : (D')^n \to R^n$ and $r \in R^n$. We have the following

Theorem 1. Problem (3), (4) has a unique distributional solution for any $r \in R^n$, if and only if the corresponding homogeneous linear system

$$x' = Ax \tag{5}$$

with the boundary value condition

$$Lx = 0 \tag{6}$$

has only the trivial solution $x = 0$.

Proof. If U is the fundamental matrix of solutions of system (5), $U'(t) = A(t)U(t)$, then the general solution x of system (3) is of the form

$$x = Uc + Ug\,, \tag{7}$$

where g is a primitive of the distribution $U^{-1}f$ and c is the constant distribution corresponding to the vector $c \in R^n$. Thus equation (4) may be written in the form $L[Uc + Ug] = r$, equivalent by (2) to the system of n linear algebraic equation with unknown c,

$$[L^n U]c = r - L[Ug]\,. \tag{8}$$

This system has a solution for any $r \in R^n$, if and only if

$$\det L^n U \neq 0 \tag{9}$$

i.e., if and only if the associated homogeneous problem (5), (6) has only the trivial solution. Furthermore, if condition (9) is satisfied, then the solution x of problem (3), (4) is unique and by (7), (8) is given by the formula

$$x = U[L^n U]^{-1}(r - L[Ug]) + Ug .$$

Remark. From the reflexivity of space D it follows that if $L : (D')^n \to R^n$ is a linear continuous mapping, then there exists a matrix function $\{\varphi_{ij}\} \in (D)^{n\times n}$ such that

$$Lx = \begin{bmatrix} \sum_{i=1}^{n} \langle x_i, \varphi_{1i} \rangle \\ \cdots\cdots \\ \sum_{i=1}^{n} \langle x_i, \varphi_{ni} \rangle \end{bmatrix}$$

for each $x = (x_1, \ldots, x_n) \in (D')^n$.

A simple corollary of Theorem 1 is

Theorem 2. Let $a_i \in C^\infty$, $\varphi_i \in D$, $i = 1, \ldots, n$, $h \in D'$. Then the equation

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = h \tag{10}$$

has for any $r_i \in R$ exactly one solution satisfying the conditions

$$\langle x, \varphi_i \rangle = r_i, \quad i = 1, \ldots, n , \tag{11}$$

if and only if $x = 0$ is the unique solution of the equation

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = 0 \tag{12}$$

satisfying the conditions

$$\langle x, \varphi_i \rangle = 0 , \quad i = 1, \ldots, n . \tag{13}$$

APPLICATIONS

1° In 1947 M. Biernacki proved [1], that if the characteristic equation

$$v^n + b_1 v^{n-1} + \ldots + b_{n-1} v + b_n = 0 \quad (b_i \in R) , \tag{14}$$

has only real roots, then for the homogeneous linear equation

$$x^{(n)} + b_1 x^{(n-1)} + \ldots + b_n x = 0$$

the boundary value problem

$$x(t_i) = r_i,\ t_1 < t_2 < \ldots < t_n$$

has only the trivial solution $x(t) \equiv 0$ for any $r_i \in R$. Using this result we have the following.

Theorem 3. Let $a_i(t) \equiv b_i \in R$, $\varphi_i \in D_+$, $i = 1, \ldots, n$ satisfy the conditions $\varphi_i(t)\cdot\varphi_j(t) \equiv 0$ for $i \neq j$. If the characteristic equation (14) has only real roots, then for any distribution $h \in D'$ and any $r_i \in R$, $i = 1, 2, \ldots, n$, the boundary value problem (10), (11) has exactly one solution.

Proof. By Theorem 2, it is enough to show under our assumptions that the homogeneous problem (12), (13) has only the solution $x = 0$. Let x be a non-trivial solution of problem (12), (13). By the mean-value theorem

$$0 = \langle x, \varphi_i \rangle = \int_{\Delta\varphi_i} x(s)\varphi_i(s)\,ds = x(t_i) \int_{\Delta\varphi_i} \varphi_i(s)\,ds = x(t_i)\,,$$

where $t_i \in \Delta\varphi_i$, $i = 1, 2, \ldots, n$. Hence and from the assumption $\varphi_i(t)\cdot\varphi_j(t) \equiv 0$ for $i \neq j$, there is a sequence $t_1 < t_2 < \ldots < t_n$ such that $x(t_i) = 0$, $i = 1, \ldots, n$, but this contradicts the result of Biernacki.

2° Developing the idea of de la Vallée Poussin, A. Levine has proved [4], [5] that if coefficients $a_i \in C^\infty$, satisfy one of the inequalities

$$\sum_{k=1}^{n-1} \frac{(\beta-\alpha)^k}{k!} \max_{[\alpha,\beta]} |a_k(t)| + \frac{(n-1)^{n-1}}{n^n n!} (\beta-\alpha)^n \max_{[\alpha,\beta]} |a_n(t)| < 1\,, \tag{15}$$

or

$$\sum_{k=1}^{n} \frac{(\beta-\alpha)^k}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} \max_{[\alpha,\beta]} |a_k(t)| < 1 \tag{16}$$

($[z]$ denotes the whole part of z), then the function $x = 0$ is the unique solution of the boundary problem

$$x^{(n)} + a_1(t)x^{(n-1)} + \ldots + a_n(t)x = 0\,,$$

and

$$x(t_i) = 0\,, \quad t_1 < t_2 < \ldots < t_n\,, \quad t_i \in [\alpha, \beta]\,.$$

Theorem 4. Let functions $a_i \in C^\infty$ satisfy on the interval $[\alpha, \beta]$ one of the inequalities, (15) or (16), and let the functions $\varphi_i \in D_+$, $i = 1, \ldots, n$ be such that $\operatorname{supp} \varphi_i \subset [\alpha, \beta]$, $\varphi_i(t)\cdot\varphi_j(t) \equiv 0$ for $i \neq j$. Then for any distribution $h \in D'$, and any values $r_i \in R$, $i = 1, \ldots, n$ there exists exactly one solution of boundary problem (10), (11).

The proof will be omitted as it is similar to that of Theorem 3.

For any distribution $f \in (D')^n$, let f_p denote the distribution defined by the formula

$$\langle f_p, \varphi\rangle = \langle f, \varphi \circ \tau_p\rangle , \quad \varphi \in D , \quad \tau_p(t) = t+p .$$

We say that $f \in (D')^n$ is a distribution of period p if $f_p = f$. It is easy to verify that for any $f \in (D')^n$

(17) $$(f_p)' = (f')_p .$$

If $A \in (C^\infty)^{n\times n}$ is a matrix function of period p, then a fundamental matrix of solution U of the system $x' = Ax$ has a representation of the form

$$U(t) = Z(t)e^{St} , \quad \text{where} \quad Z(t+p) \equiv Z(t)$$

and S is a constant matrix. Hence, for any distribution $g \in (D')^n$ we have

(18) $$(Ug)_p = Ue^{-Sp}g_p , \quad (U^{-1}g)_p = e^{Sp}U^{-1}g_p .$$

We shall show the following

Theorem 5. Let $A \in (C^\infty)^{n\times n}$ be a matrix function of period p. The equation

(19) $$x' = Ax+f$$

has the unique solution of period p for any distribution $f \in (D')^n$ of period p, if and only if the trivial distribution $x = 0$ is the unique solution of period p of the corresponding homogeneous system

(20) $$x' = Ax .$$

Proof. The family of all solutions of equation (19) is given by the explicit formula

$$x = Uc + Ug ,$$

where g is a primitive of the distribution $U^{-1}f$ and c is the constant distribution corresponding to a vector $c \in R^n$. Hence, x is a solution of period p of (19), if and only if

$$x = Uc+Ug = (Uc)_p + (Ug)_p = x_p$$

and then by the formula (18)

(21) $$U(I-e^{-Sp})c = U(e^{-Sp}g_p - g) .$$

The distribution

(22) $$c_g = e^{-Sp}g_p - g$$

is the constant distribution. Indeed, by (17), (18) and the definition of g we have

$$(c_g)' = e^{-Sp}(g_p)' - g' = e^{-Sp}(g')_p - g' = e^{-Sp}(U^{-1}f)_p - U^{-1}f$$
$$= U^{-1}f_p - U^{-1}f = 0 .$$

Thus from (21) and (22) the equation (19) has a solution of period p, if and only if the algebraic equation

$$(I - e^{-Sp})c = c_g \tag{23}$$

has a solution.

Assume now that $x = 0$ is the unique solution of period p of the homogeneous system (20). This fact is equivalent to the algebraic system

$$(I - e^{-Sp})c = 0$$

has the unique solution $c = 0$, and hence implies the existence of the unique solution of (23) for any vector $c_g \in R^n$. Counting this value of c_g in (23) gives the explicit formula

$$x = U(I - e^{-Sp})^{-1}(e^{-Sp}g_p - g) + Ug ,$$

where $g' = U^{-1}f$.

Definition. We shall say that a distribution $y \in (D')^n$ is bounded if, for every $\varphi \in D$ the function

$$v(h) = \langle y_h, \varphi\rangle = \langle y, \varphi \circ \tau_h\rangle$$

is bounded [6].

Theorem 6. Let $A \in (C^\infty)^{n\times n}$ and $f \in (D')^n$ be of period p. Then the equation (19) has a solution of period p, if and only if it has at least one bounded solution.

Proof. If x is a solution of period p of equation (19), then for any $\varphi \in D$ the function $v(h) = \langle x, \varphi \circ \tau_h\rangle$ is continuous and periodic of period p and thus is bounded. Hence, according to the definition, x is the bounded distribution.

In order to prove the converse, assume that the equation (19) has a bounded solution $y \in (D')^n$. Let $b \in R^n$ is such that $y = Ub + Ug$, where U is a fundamental matrix of (20), and $g' = U^{-1}f$. By (18) $y_p = (Ub)_p + (Ug)_p = Ue^{-Sp}b + Ue^{-Sp}g_p$.
As we have shown in the proof of the preceding theorem, the distribution c_g given by (22) is constant. Therefore we have

$$y_p = U(e^{-Sp}b + c_g) + Ug ,$$

and by induction

$$y_{mp} \pm U(e^{-mSp}b + \sum_{k=0}^{m-1} e^{-kSp}c_g) + Ug \tag{24}$$

for every integer m.

Let (α, β) be an interval on R, such that the matrix

$$\begin{bmatrix} u_{11}(s_{11}), \dots, u_{1n}(s_{1n}) \\ \dots \quad \dots \quad \dots \\ u_{n1}(s_{n1}), \dots, u_{nn}(s_{nn}) \end{bmatrix}$$

is non singular for any $s_{11}, \dots, s_{nn} \in (\alpha, \beta)(U = \{u_{ij}\})$. Let φ be any element of D_+ with support in (α, β). Denote by d_m the vector of R^n by putting

$$d_m = e^{-mSp}b + \sum_{k=0}^{m-1} e^{-kSp}c_g, \quad m = 1, 2, \dots \tag{25}$$

From (24) we have the following

$$\langle y, \varphi \circ \tau_p^m \rangle = \langle y_{mp}, \varphi \rangle = \langle Ud_m, \varphi \rangle + \langle Ug, \varphi \rangle, \quad m = 1, 2, \dots \tag{26}$$

The assumption that y is the bounded distribution implies that the sequence $\langle y, \varphi \circ \tau_p^m \rangle$ is bounded and hence, by (26) the sequence

$$l_m = \langle Ud_m, \varphi \rangle, \quad m = 1, 2, \dots$$

is bounded too. We have

$$l_m = \int_{\Delta\varphi} \varphi(t)\, U(t)\, d_m\, dt = Hd_m, \quad \text{where} \quad H = \int_{\Delta\varphi} \varphi(t)\, U(t)\, dt\,. \tag{27}$$

The choice of (α, β) and φ implies the non singularity of matrix H. Now, by (27) the sequence d_m is also bounded.

As we have seen in the proof of Theorem 5, the equation (19) has a solution of period p, if any only if the system of algebraic linear equations (23) has a solution. Suppose it possible that equation (23) has no solution. Then there exists a vector c_0 such that

$$(I - e^{-Sp})^* c_0 = 0 \quad \text{and} \quad (c_0, c_g) \neq 0\,,$$

($(\cdot, \cdot)$ denote the scalar product). Thus $c_0 = (e^{-kSp})^* c_0$ for $k = 0, 1, \dots$ Multiply equation (25) scalarly by c_0 to obtain

$$(d_m, c_0) = (e^{-mSp}b, c_0) + \Big(\sum_{k=0}^{m-1} e^{-kSp}c_g, c_0\Big) = \big(b, (e^{-mSp})^* c_0\big) +$$

$$+ \Big(c_g, \sum_{k=0}^{m-1} (e^{-kSp})^* c_0\Big) = (b, c_0) + m(c_g, c_0)\,, \quad m = 1, 2, \dots$$

As $(c_g, c_0) \neq 0$, the sequence (d_m, c_0) is not bounded. But this is impossible since the sequence d_m is bounded, and therefore equation (23) has a solution. This proves the theorem.

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