ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO

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Maria Mazurek

Continuation of Separately Analytic Functions

Abstract. Let E and F be compact subsets of C^n and C^m, respectively, such that $\Phi(z, E)$ and $\Phi(w, F)$ (def. see (1)) are continuous in C^n and C^m, respectively. Given $R > 1$, $\varrho > 1$, define

$$\begin{aligned} &D = \{z \in C^n : \Phi(z, E) < R\}, \\ \text{(A.1)}\qquad &G = \{w \in C^m : \Phi(w, F) < \varrho\}, \\ &X = (D \times F) \cup (E \times G). \end{aligned}$$

We shall prove that if $f: X \to C$ is locally bounded and separately analytic (def. 1.2) then f is continuable to an analytic function f in the domain of holomorphy $\Omega(X)$ given by

$$\Omega(X) = \left\{(z, w) \in C^{n+m} : \frac{\log\Phi(z, E)}{\log R} + \frac{\log\Phi(w, F)}{\log\varrho} < 1\right\}.$$

In the case when E and F are closed Weil's polyhedrons (def. [5]) defined by homogeneous polynomials the assumption of the boundedness is superfluous.

By the way we prove that $\dfrac{\log\Phi(w, F)}{\log\varrho}$ is equal to function $h_G(w, F)$ defined in Theorem 1.3.

Next we shall prove:

If D is a bounded circular domain of holomorphy the necessary and sufficient condition that $\Phi(z, \bar{D}) = \Phi(z, \partial D)$ be continuous in C^n is for any $a \in \partial K(0, 1)$, $\partial D \cap \pi_a$ ($\pi_a = \{z \in C^n : z = \lambda a, \lambda \in C\}$) be a circle.

We also give the example of a circular bounded domain of holomorphy where there exists $a \in \partial K(0, 1)$ such that $\partial D \cap \pi_a$ is a ring.

1. INTRODUCTION

The following definitions and theorems are the basic tools in the proofs given in this paper.

Let E be a compact subset of the space C^n of n-complex variables. Let $p^{(\nu)} = \{p_1, \ldots, p_\nu\}$, where $p_i = \{z_{i_1}, \ldots, z_{i_n}\}$ $i = 1, 2, \ldots, \nu^*$, $\nu^* = \frac{(\nu+n)!}{n!\nu!}$ be a system of points such that $p_i \in E$ and the determinant

$$V(p^{(\nu)}) = \det[z_{i_1}^{k_1} \ldots z_{i_n}^{k_{l_n}}] \qquad i, l = 1, 2, \ldots, \nu^*$$

is different from zero.

A system $\{\eta_1, \ldots, \eta_{\nu^*}\} = \eta^{(\nu)}$ will be called the νth extremal system of E if

$$|V(p^{(\nu)})| \leqslant |V(\eta^{(\nu)})|$$

for every system points $p^{(\nu)}$ of E.

Let us denote by $V_i(p^{(\nu)}, z)$ the determinant formed for the system of points $\{p_1, \ldots, p_{i-1}, z, p_{i+1}, \ldots, p_{\nu^*}\}$ and let

$$L^i(z, p^{(\nu)}) = \frac{V_i(z, p^{(\nu)})}{V(p^{(\nu)})}.$$

If $f: E \to C$ then we put

$$\|f\|_E = \sup\{|f(z)| : z \in E\}.$$

Let us denote by $F(E)$ the set of all polynomials p such that $\|p\|_E \leqslant 1$.

Definition 1.1. ([2]). The extremal function of the compact set E,

$$\Phi(z, E) = \sup\{|p(z)|^{\frac{1}{\deg p}} : p \in F(E)\}.$$

The following theorems are known:

Theorem 1.1 ([2]). If the complement of a compact set E is connected, $\Phi(z, E)$ is continuous in C^n and the function f is holomorphic in a neighbourhood of E, then the sequence of interpolation polynomials

$$L_\nu(z, f) = \sum_{i=1}^{\nu^*} f(\eta_i) L^i(z, \eta^{(\nu)}) \qquad \nu = 1, 2, \ldots$$

converges maximally to f (def. 1.4).

Definition 1.2 ([3]). A set $E \in (L)$ if and only if for any $a \in E$, any $r > 0$, $\varepsilon > 0$ and for every family of polynomials p of n-complex variables such that:

$$M_F(z) = \sup\{|p(z)| : p \in F\} < \infty, \qquad z \in E \cap K(a, r)$$

there exist two positive numbers $M = M(a, \varepsilon)$ and $\delta = \delta(a, \varepsilon)$ such that

$$|p(z)| \leqslant M \exp(\varepsilon \deg p), \ \|z - a\| < \delta, \ p \in F.$$

Theorem 1.2 ([3]). Assume that: a) G is an open set in C^n, b) E is a compact subset of G, $E \in (L)$, c) $\{\lambda_\nu\}$ is a sequence of positive real numbers, d) T is an arbitrary non-empty set, and e) for every $t \in T$, $\{f_\nu(z, t)\}$ is a sequence of analytic functions in G such that

i) $\sup_{t\in T} \frac{1}{\lambda_\nu} \log |f_\nu(z, t)| \leqslant K = \text{const.}, \ z \in G, \ \nu \geqslant 1$

ii) $\limsup_{\nu\to\infty} \sup_{t\in T} \frac{1}{\lambda_\nu} \log |f_\nu(z, t)| \leqslant A = \text{const.}, \ z \in E.$

Then for every $\varepsilon > 0$ there exist a positive number $M = M(\varepsilon)$ and an open subset $U = U(\varepsilon)$ of G such that $E \subset U$ and

(iii) $|f_\nu(z, t)| \leqslant M \exp\big((A+\varepsilon)\lambda_\nu\big), \ z \in U, \ t \in T, \ \nu \geqslant 1.$

Theorem 1.3 ([3]). Let G be a domain in the space C^n and F be a compact subset of G. Let

$\mathcal{M} = \mathcal{M}(G, F) = \{u : u$ is a plurisubharmonic function in G $u|_F \leqslant 0, \ u|_G \leqslant 1$

and

$$(1.3) \qquad h_G(w, F) = \limsup_{w'\to w} \sup \{u(w') : u \in \mathcal{M}\}, \qquad w \in G.$$

Then $h_G(w, F)$ is a plurisubharmonic function in G. Moreover for any plurisubharmonic function v in G such that $v|_F \leqslant m$, $v|_G \leqslant M$ we have

$$(1.4) \qquad v(w) \leqslant m + (M-m) h_G(w, F), \qquad w \in G.$$

Definition 1.3 ([3]). A function f is separately analytic on

$$X = (D \times F) \cup (E \times G)$$

if and only if f is analytic on D for every $w \in F$ and f is analytic on G for every $z \in E$.

Definition 1.4 ([2]). We say that the sequence of polynomials $\{P_\nu\}$, $\deg P_\nu = \nu$ converges maximally to the function f holomorphic in D (def. (A.1)) and uncontinuable on D' for $R' > R$ if and only if

$$\limsup_{\nu\to\infty} \big(\max_{z\in E} |f(z) - P_\nu(z)|\big)^{1/\nu} = \frac{j}{R}.$$

It has been proved in [2] that for any $z \in C^n$

$$(1.1) \qquad \lim_{\nu\to\infty} \max_{i=1\ldots\nu^*} |L^i(z, \eta^{(\nu)})|^{1/\nu} = \Phi(z, E)$$

and

$$(1.2) \qquad |p(z)| \leqslant \|p\|_E \Phi_{(z,E)}^{\deg p} \text{ for any polynomial } p.$$

2. THE ENVELOPE OF HOLOMORPHY OF THE UNION OF TWO DOMAINS OF HOLOMORPHY DEFINED BY EXTREMAL FUNCTIONS OF CERTAIN COMPACT SETS

Theorem 2.1. Let $E \subset C^m$ be a compact set such that $C^m \setminus E$ is connected and $\Phi(z, E)$ is continuous in C^m. Let $F \subset C^m$ be a compact set such that $C^m \setminus F$ is connected and $\Phi(w, F)$ is continuous in C^m. If f is a bounded and separately analytic function in

$$X = (D \times F) \cup (E \times G)$$

where

$$D = \{z \in C^m : \Phi(z, E) < R\}, \quad G = \{w \in C^m : \Phi(w, F) < R\}, \quad R > 1,$$

then f is continuable to a holomorphic function in the domain of holomorphy $\Omega(X)$ given by

$$\Omega(X) = \left\{(z, w) \in C^{m+m} : \frac{\log \Phi(z, E)}{\log R} + \frac{\log \Phi(w, F)}{\log R} < 1\right\}.$$

Proof: Let $M > 0$ be such that $|f(z, w)| \leqslant M$ on X. Let us denote:
$\{\eta^{(\nu)}\}$ — the νth extremal system of points of E
$\{\gamma^{(\mu)}\}$ — the μth extremal system of points of F.

(2.1)
$$P_\nu(z, w) = \sum_{i=1}^{\nu^*} f(\eta_i, w) L^i(z, \eta^{(\nu)})$$

is the corresponding interpolating polynomials of $z \in C^m$.

(2.2)
$$Q_\mu(z, w) = \sum_{j=1}^{\mu^*} f(z, \gamma_j) L^j(w, \gamma^{(\mu)})$$

is the corresponding interpolating polynomials of $w \in C^m$.

Take the sequence of the functions

$$\tilde{f}_\nu(z, w) = f(z, w) - P_\nu(z, w).$$

For every $z \in E$ $\tilde{f}_\nu(z, w)$ is analytic in G and

i) $\max\limits_{z \in E} \frac{1}{\nu} \log |\tilde{f}_\nu(z, w)| \leqslant \frac{1}{\nu} \log M + \frac{1}{\nu} \log(1 + \nu^*) \leqslant K$, $w \in G$, $\nu \geqslant 1$

ii) $\limsup\limits_{\nu \to \infty} \max\limits_{z \in E} \frac{1}{\nu} \log |\tilde{f}_\nu(z, w)| \leqslant \log \frac{1}{R}$, $w \in F$.

The second inequality follows from the proof of Theorem 1.1. Thus by Theorem 1.2 for every R_1 such that $1 < R_1 < R$ there exists M_1 such that

(2.3)
$$\max_{E \times F} |f - P_\nu| \leqslant \frac{M_1}{R_1^\nu}, \quad \nu \geqslant 1.$$

By analogy we can prove that

$$\max_{E\times F}|f-Q_\mu| \leqslant \frac{M_2}{R_1^\mu}, \quad \mu \geqslant 1. \tag{2.4}$$

Taking if necessary $\max\{M_1, M_2, M\}$ we can assume that $M_1 = M_2 = M$. Setting

$$L_{\nu\mu}(z, w) = \sum_{i=1}^{\nu^*}\sum_{j=1}^{\mu^*} f(\eta_i, \gamma_j) L^i(z, \eta^{(\nu)}) L^j(w, \gamma^{(\mu)})$$

the following estimations are obtained:

$$\max_{E\times F}|P_\nu(z, w) - L_{\nu\mu}(z, w)| = \max_{E\times F}\left|\sum_{i=1}^{\nu^*}\left[f(\eta_i, w) - \sum_{j=1}^{\mu^*} f(\eta_i, \gamma_j) L^j(w, \gamma^\mu)\right] L^i(z, \eta^{(\nu)})\right| \leqslant \frac{\nu^* M}{R_1^\mu} \tag{2.5}$$

$$\max_{E\times F}|Q_\mu(z, w) - L_{\nu\mu}(z, w)| \leqslant \frac{\mu^* M}{R_1^\nu}, \quad \nu, \mu \geqslant 1. \tag{2.6}$$

Hence

$$\begin{gathered}\max_{E\times F}|L_{\sigma,\sigma+1} - L_{\sigma,\sigma}| \leqslant \frac{2M\sigma^*}{R_1^\sigma}, \quad \max_{E\times F}|L_{\sigma+1,\sigma+1} - L_{\sigma,\sigma+1}| \leqslant \frac{2M(\sigma+1)^*}{R_1^\sigma} \\ \max_{E\times F}|L_{\sigma+1,\sigma} - L_{\sigma,\sigma}| \leqslant \frac{2M\sigma^*}{R_1^\sigma}, \quad \max_{E\times F}|L_{\sigma+1,\sigma+1} - L_{\sigma+1,\sigma}| \leqslant \frac{2M(\sigma+1)}{R_1^\sigma}.\end{gathered} \tag{2.7}$$

It follows from (2.1)—(2.6) that $L_{\sigma\sigma} \Rightarrow f$ on $E\times F$. From (2.7) we get

$$\max_{E\times F}|L_{\sigma+1,\sigma+1} - L_{\sigma,\sigma}| \leqslant \frac{4M(\sigma+1)^*}{R_1^\sigma}.$$

Since by (1.2) for any $(z, w) \in C^{n+m}$ we have

$$|L_{\sigma+1,\sigma+1}(z, w) - L_{\sigma,\sigma}(z, w)| \leqslant \frac{4M(\sigma+1)^*}{R_1^\sigma}\Phi_{(z,E)}^{\sigma+1}\Phi_{(w,F)}^{\sigma+1}$$

the series

$$\sum_{\sigma=1}^{\infty}[L_{\sigma+1,\sigma+1} - L_{\sigma,\sigma}] + L_{11} \tag{2.8}$$

is uniformly convergent in an arbitrary set

$$\Omega R_2 = \{(z, w) \in C^{n+m} : \Phi(z, E)\Phi(w, F) \leqslant R_2\}, \quad R_2 < R_1.$$

Therefore the series (2.8) is convergent uniformly on every compact subset of the set

$$\Omega(X) = \{(z, w) \in C^{n+m} : \Phi(z, E)\Phi(w, F) < R\}.$$

Since $\log \frac{\Phi(z, E)\Phi(w, F)}{R}$ is a plurisubharmonic function ([2]), so $\Omega(X)$ is a domain of holomorphy. Observe that $\Omega(X)$ can be written in the form

$$\Omega(X) = \left\{(z, w) \in C^{n+m} : \frac{\log\Phi(z, E)}{\log R} + \frac{\log\Phi(w, F)}{\log R} < 1\right\},$$

Moreover, for every $w \in F$

$$\tilde{f}(z, w) = \sum_{\sigma=1}^{\infty} [L_{\sigma+1,\sigma+1}(z, w) - L_{\sigma,\sigma}(z, w)] + L_1(z, w) = f(z, w), \quad z \in E.$$

Since $\Phi(z, E)$ is continuous in C^n, so $E \in (L)$. Thus $\tilde{f}(z, w) = f(z, w)$ on D ([3]). In the same way we get for every $z \in E$

$$\tilde{f}(z, w) = f(z, w) \quad \text{on} \quad G.$$

So we obtain

$$\tilde{f}(z, w) = f(z, w) \quad \text{on} \quad X.$$

Q.E.D.

In the last theorem it suffices to assume that the function $f(z, w)$ is locally bounded. Indeed, then $f(z, w)$ is bounded on

$$X_k = (D_k \times F) \cup (E \times G_k) \qquad k = 1, 2, \ldots$$

where

$$D_k = \left\{z \in C^n : \Phi(z, E) < \frac{kR}{k+1}\right\}, \qquad G_k = \left\{w \in C^m : \Phi(w, F) < \frac{kR}{k+1}\right\},$$

so the function f is continuable to the function $\tilde{f}_k$ holomorphic in

$$\Omega(X_k) = \left\{(z, w) \in C^{n+m} : \Phi(z, E)\Phi(w, F) < \frac{kR}{k+1}\right\}.$$

The set $\Omega(X) = \bigcup_{k=1}^{\infty} \Omega(X_k)$. Observe that the function $\tilde{f}$ defined by

$$\tilde{f}(z, w) = \tilde{f}_k(z, w) \quad \text{for} \quad (z, w) \in \Omega(X_k)$$

is the required function.

Keeping the notation of Theorem 2.1 we have

Theorem 2.1′. If f is separately analytic and locally bounded in

$$X = (D \times F) \cup (E \times G)$$

then f can be holomorphy continued to

$$\Omega(X) = \left\{(z, w) \in C^{n+m} : \frac{\log\Phi(z, E)}{\log R} + \frac{\log\Phi(w, F)}{\log R} < 1\right\}.$$

Theorem 2.2. If F and G fulfil the same conditions as in Theorem 2.1, then

$$\frac{\log\Phi(w,F)}{\log R} = h_G(w,F).$$

Proof: Let E and D be the same as in Theorem 2.1 and let f be separately analytic and bounded on

$$X = (D\times F)\cup(E\times G).$$

There exists $\bar{M}>0$ such that $|f(z,w)|\leqslant \bar{M}$ for $(z,w)\in X$.

The series (2.1) is convergent uniformly on every compact subset of the set

$$\Omega_1(X) = \left\{(z,w)\in C^{n+m}: \frac{\log\Phi(z,E)}{\log R}+h_G(w,F)<1\right\}.$$

Let S be a compact subset of $\Omega_1(X)$. Since

$$\frac{\log\Phi(z,E)}{\log R}+h_G(w,F)$$

is a plurisubharmonic function, it follows that there exist a and a' such that

$$\max_S\left[\frac{\log\Phi(z,E)}{\log R}+h_G(w,F)\right]\leqslant a'<a<1.$$

Let $(z_0,w_0)\in S$

$$\frac{\log\Phi(z_0,E)}{\log R}+h_G(w_0,F)\leqslant a'<a<1.$$

Put $\Phi(z_0,E)=R_2$. There exists R_1 such that $R_2<R_1<R$ and

(2.9) $$\frac{\log R_2}{\log R_1}<a-h_G(w_0,F).$$

From (2.3) it follows that

$$\max_{E\times F}|P_{\nu+1}-P_\nu|\leqslant\frac{2M}{R_1^\nu},\qquad M=M(R_1).$$

We can assume that $M\geqslant\bar{M}$. Moreover

$$\max_{\bar{D}_{R_2}\times F}|P_{\nu+1}-P_\nu|\leqslant 2MR_2\left(\frac{R_2}{R_1}\right)^\nu,$$

where

$$[D_{R_2}=\{z\in C^n:\Phi(z,E)<R_2\}$$

and

$$\sup_{\bar{D}_{R_2}\times G}|P_{\nu+1}-P_\nu|\leqslant\sup_{\bar{D}_{R_2}\times G}|P_{\nu+1}|+\sup_{\bar{D}_{R_2}\times G}|P_\nu|\leqslant 2M(\nu+1)^*R_2^{\nu+1}.$$

Applying Theorem 1.3 to the functions

$$u_\nu = \log\frac{|P_{\nu+1}-P_\nu|}{(\nu+1)^* 2MR_2}$$

we get

$$|P_{\nu+1}(z,w)-P_\nu(z,w)| \leqslant 2MR_2(\nu+1)^*\exp\left(\nu\log R_1\left(\frac{\log R_2}{\log R_1}+h_G(w,F)-1\right)\right)$$

for $(z,w)\in \bar{D}_{R_2}\times G$.
So for (z_0, w_0) we have the estimation:

$$(2.10)\qquad |P_{\nu+1}(z_0,w_0)-P_\nu(z_0,w_0)| \leqslant 2MR_2(\nu+1)^*\exp[\nu\log R_1(a-1)].$$

For every $(z_0,w_0)\in S$, $R_2\leqslant R^{a'}$, thus we can take R_1 such that it fulfils (2.9) and $R^{a'}<R_1$. Hence we get the estimation:

$$(2.11)\qquad \max_S |P_{\nu+1}-P_\nu| \leqslant 2MR^{a'}(\nu+1)^*\exp[\nu a'(a-1)\log R],$$

so the series (2.1) is convergent uniformly on S.

The function

$$\tilde{f}(z,w) = \sum_{\nu=1}^{\infty}[P_{\nu+1}(z,w)-P_\nu(z,w)]+P_1(z,w)$$

equals $f(z,w)$ on X.
By (2.3)

$$\tilde{f}(z,w) = f(z,w)\qquad \text{on}\qquad E\times F.$$

For any $w\in F$ $\tilde{f}(z,w)=f(z,w)$ for $z\in E$, since $E\in(L)$ so $\tilde{f}(z,w)=f(z,w)$ for $z\in D$ ([5]).
By analogy for any $z\in E$ $\tilde{f}(z,w)=f(z,w)$ for $w\in G$, thus $\tilde{f}(z,w)=f(z,w)$ for $(z,w)\in X$.

Since

$$\frac{\log\Phi(w,F)}{\log R}\leqslant h_G(w,F)\ \text{on}\ G,\ \text{so}\ \Omega_1(X)\subset\Omega(X).$$

It suffices to prove that $\Omega_1(X)\supset\Omega(X)$. $\Omega_1(X)$ is a domain of holomorphy, thus there exists the function g holomorphic in $\Omega_1(X)$ and uncontinuable.

Since $X\subset\Omega_1(X)$, then g fulfils the assumptions of Theorem 2.1′. This implies the required inclusion.

Suppose that there exist w_0 and $\partial>0$ such that

$$h_G(w_0,F) = \partial+\frac{\log\Phi(w_0,F)}{\log R}.$$

Then the point (z_0, w_0) such that

$$\frac{\log \Phi(z_0, E)}{\log R} + h_G(w_0, F) = 1$$

belongs to $C^{n+m} \setminus \Omega_1(X)$ and on the other hand belongs to $\Omega(X)$; so we have the contradiction and hence

$$h_G(w, F) = \frac{\log \Phi(w, F)}{\log R}$$

Q.E.D.

Theorem 2.3. If D, E, F fulfil the conditions of Theorem 2.1 and

$$G = \{w \in C^m : \Phi(w, F) < \varrho\}, \quad \varrho > 1$$

then every function which is separately analytic and locally bounded on

$$X = (D \times F) \cup (E \times G)$$

is continuable to a holomorphic function in

$$\Omega(X) = \left\{(z, w) \in C^{n+m} : \frac{\log \Phi(z, E)}{\log R} + \frac{\log \Phi(w, F)}{\log \varrho} < 1\right\}.$$

Proof: Observe that f is bounded on

$$X_k = (D_k \times F) \cup (E \times G_k) \quad k = 1, 2, \ldots$$

where

$$D_k = \{z \in C^n : \Phi(z, E) < R^{\frac{k}{k+1}}\}, \quad G_k = \{w \in C^m : \Phi(w, F) < \varrho^{\frac{k}{k+1}}\}$$

and

$$h_{G_k}(w, F) = \frac{k+1}{k} h_G(w, F).$$

The function f is continuable to the function $\tilde{f}_k$ holomorphic in

$$\Omega(X_k) = \left\{(z, w) \in C^{n+m} : \frac{\log \Phi(z, E)}{\log R} + \frac{\log \Phi(w, F)}{\log \varrho} < \frac{k}{k+1}\right\}$$

and hence (reasoning as in the proof of Theorem 2.1′) the thesis is obtained.

3. THE EXTREMAL FUNCTION FOR COMPACT CIRCULAR SETS

It has been proved in [2] that the extremal function for a compact circular set E is equal to $\max\{1, \psi(z, E)\}$ where

$$\psi(z, E) = \max_{p \in F_0(E)} |p(z)|^{\frac{1}{\deg p}}$$

and

$$F_0(E) = \{p : p \text{ is a homogeneous polynomial such that } \|p\|_E \leqslant 1\}.$$

The set E is called circular if along with the point $z^0 = (z_1^0, \dots, z_n^0)$, $z^0 \in E$ all the points of the circle

$$z = e^{i\theta} z^0 = (e^{i\theta} z_1^0, \dots, e^{i\theta} z_n^0), \quad 0 \leqslant \theta \leqslant 2\pi$$

belong to E.

Proposition 3.1. Let p_i $i = 1, 2, \dots, r$ be homogeneous polynomials in C^n, $\deg p_i = \alpha_i$ and let

$$E = \{z \in C^n : |p_i(z)| \leqslant 1, \ i = 1, \dots, r\}.$$

Then

1. int $E = \overset{0}{E}$ is connected and $\overset{0}{E} = \{z \in C^n : \max\limits_{i=1\dots r} |p_i(z)| < 1\}$.

2. If E is compact then $\psi(z, E) = \max\limits_{i=1\dots r} |p_i(z)|^{\frac{1}{\alpha_i}}$.

Proof: Let us denote

$$\max_{i=1\dots r} |p_i(z)|^{\frac{1}{\alpha_i}} = T(z, E).$$

Since $E_i = \{z \in C^n : |p_i(z)| \leqslant 1\}$, $i = 1, 2, \dots, r$ is a starlike set, therefore the set $E = \prod\limits_{i=1}^{r} E_i$ is also starlike and hence must be connected.

Because $T(z, E)$ is a continuous function, then

$$\overset{0}{E} = \{z \in C^n : T(z, E) < 1\}, \quad \partial E = \{z \in C^n : T(z, E) = 1\}.$$

Since $\psi(z, E) \geqslant T(z, E)$ it is enough to show the inequality

$$\psi(z, E) \leqslant T(z, E) \quad \text{for} \quad z \in C^n.$$

Since E is a compact set then for every $a \in C^n$ a half-line l_a defined by

$$l_a = \{z \in C^n : z = \lambda a, \lambda \geqslant 0\}$$

has at least one point common with ∂D.

Suppose that there exists $z^0 \in C^n$ such that $T(z^0, E) < \psi(z^0, E)$ i.e. that there exists a homogeneous polynomial q such that

$$\|q\|_E \leqslant 1 \quad \text{and} \quad |q(z^0)|^{\frac{1}{\deg q}} > \max_{i=1\dots r} \{|p_i(z^0)|^{\frac{1}{\alpha_i}}\}$$

Let us take the half line l_{z^0},

$$|q(\lambda z^0)|^{\frac{1}{\deg q}} = \lambda |q(z^0)|^{\frac{1}{\deg q}} > \max_{i=1\dots r} (\lambda |p_i(z^0)|^{\frac{1}{\alpha_i}}) = \max_{i=1\dots r} \{|p_i(\lambda z^0)|^{\frac{1}{\alpha_i}}\}.$$

Since there exists λ_1 such that the point $\lambda_1 z^0 \in \partial E$ we get a contradiction. So

$$\psi(z, E) \leqslant T(z, E)$$

as asserted.

Theorem 3.1. If D is a bounded circular domain of holomorphy such that $\partial D \cap \pi_a (\pi_a = \{z \in C^n : z = \lambda a, \lambda \in C\})$ is a circle for every $a \in \partial K(0, 1)$ and if $E = \bar{D}$ or $E = \partial D$ then the extremal function of E is continuous in C^n.

Proof. $D = \operatorname{int} \bar{D}$ ([4]). Because D is convex with respect to the family of homogeneous polynomials, so there exists a sequence $\{D_\nu\}$ of Weil's polyhedrons defined by homogeneous polynomials such that

$$D = \bigcup_{\nu=1}^{\infty} D_\nu,\ D_1 \subset D_2 \subset \ldots \subset D, \quad \bar{D}_\nu = E_\nu \quad \text{and} \quad \bigcup_{z^0 \in E_\nu} K\left(z^0, \frac{1}{\nu}\right) \supset \bar{D}$$

where

$$K(x, r) = \{z \in C^n : \|z - x\| < r\},$$

for a sufficiently large ν.

Let us take a sufficiently large ν. Then the set

$$U = D \setminus \bigcup_{\varrho \in \partial D} K\left(\varrho, \frac{1}{\nu+1}\right)$$

is compact and

$$U \subset\subset D \quad \text{so} \quad \hat{U} \subset\subset D$$

where $\hat{U}$ is its convex envelope with respect to homogeneous polynomials. For any $\varrho \in \partial D$ there exists a homogeneous polynomial Q_ϱ such that

$$\|Q_\varrho(z)\|_{\hat{U}} \leqslant 1 \quad \text{and} \quad |Q_\varrho(z)|_{V_\varrho} > 1$$

where V_ϱ is a neighbourhood of ϱ.

Since ∂D is compact, then there exist homogeneous polynomials $Q_1, \ldots, Q_{p_\nu}$ such that the sets

$$E_\nu = \{z \in C^n : |Q_i(z)| \leqslant 1,\ i = 1, \ldots, p_\nu\}, \quad D_\nu = \overset{0}{E}_\nu$$

satisfy the required conditions.

First we shall show that

$$\psi(z, E) = \lim_{\nu \to \infty} \psi(z, E_\nu)$$

Let us take E_ν and a homogeneous polynomial Q such that $Q \in F_0(E_\nu)$. Then

$$\max_{z \in E} |Q(z)| \leqslant \left(1 + \frac{\frac{1}{\nu}}{m_\nu}\right)^{\deg Q} \qquad \text{where} \quad m_\nu = \min_{z \in \partial E_\nu} \|z\|.$$

Since $m_\nu \leqslant m_{\nu+1}$ and some neighbourhood of 0 belongs to D, we have

$$0 < m_{\nu_0} \leqslant m_\nu \text{ for a sufficiently large } \nu_0 \text{ and } \nu > \nu_0 .$$

Therefore

$$\max_{z \in E} |Q(z)| \leqslant \left(1 + \frac{1}{\nu m_0}\right)^{\deg Q} .$$

Hence

$$\left| \frac{Q(z)}{\left(1 + \frac{1}{\nu m_0}\right)^{\deg Q}} \right| \leqslant 1 \quad \text{for} \quad z \in E .$$

From this we get

$$\frac{\psi(z, E_\nu)}{\left(1 + \frac{1}{\nu m_0}\right)} \leqslant \psi(z, E)$$

and

$$\lim_{\nu \to \infty} \frac{\psi(z, E_\nu)}{\left(1 + \frac{1}{\nu m_0}\right)} = \lim_{\nu \to \infty} \psi(z, E_\nu) \leqslant \psi(z, E) .$$

Because the opposite inequality is obvious we have

$$\lim_{\nu \to \infty} \psi(z, E_\nu) = \psi(z, E)$$

Since $\psi(z, E)$ is lower semicontinuous in C^n and $\lim_{\nu \to \infty} \psi(z, E_\nu)$ is upper semicontinuous in C^n, so $\psi(z, E)$ is continuous in C^n.

Hence $\Phi(z, E) = \max\{1, \psi(z, E)\}$ is also continuous in C^n.

It is obvious that

$$\lim_{\nu \to \infty} \Phi(z, E_\nu) = \Phi(z, E) . \tag{3.2}$$

Theorem 3.2. If D is a bounded circular domain of holomorphy such that there exists $a \in \partial K(0, 1)$ such that $\partial D \cap \pi_a$ is a ring, then $\psi(z, \bar{D}) = \psi(z, \partial D)$ and $\Phi(z, \bar{D}) = \Phi(z, \partial D)$ are discontinuous in C^n.

Proof. Let for $a \in \partial K(0, 1)$ the ring

$$\pi_r^* = \{z \in C^n : z = \lambda a,\ 1 \leqslant |\lambda| \leqslant r\} \subset \partial D$$

The point $a \in \partial D$ so

$$\psi^*(a, \bar{D}) = \limsup_{z' \to a} \psi(z', \bar{D}) \geqslant 1 \qquad ([4])$$

Hence there exists a sequence $\{z_\nu\} \to a$ such that

$$\psi(z_\nu, \bar{D}) \to \psi^*(a, \bar{D}) \geqslant 1$$