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A Note on a Difference Inequality

1. Recently Kowalski and Pliś [1] have discussed a linear difference inequality connected with a non-linear partial differential equation of elliptic type. They have proved that the maximum of the solution does not exceed some positive quantity. This result can be used to obtain the error estimate of the difference method for a nonlinear elliptic equation.

It may be expected that in the case of a second order nonlinear ordinary differential equation the corresponding estimate and the proof become simpler than those of [1]. We will try to do this as simply as possible. In this note we deal with a linear difference inequality

(1) $$u_i^{(2)}+b_iu_i^{(1)}+c_iu_i \geqslant -\varepsilon, \qquad \varepsilon > 0$$

(here $u_i^{(2)}$ and $u_i^{(1)}$ stand for central differences of the second and first orders resp.), which is related to the ordinary differential equation. An approximation theorem will be stated elsewhere. Here we mention only that the common requirement that the differential equation in question be explicitely solved in x'' will be overcome there by using our difference inequality.

2. Fix a natural number n and set $h = \dfrac{1}{n+1}$. For a function defined on an interval, say, $[0, 1]$ we denote by u_i its value in the nodal point $x_i = ih$, $i = 0, 1, \ldots, n+1$. Define the central differences as follows

$$u_i^{(1)} = (u_{i+1}-u_{i-1}), \qquad u_i^{(2)} = (u_{i+1}-2u_i+u_{i-1}), \qquad i = 1, \ldots, n.$$

It is an immediate consequence of these definitions that

(2) $$2u_i^{(1)} = -hu_i^{(2)}+(u_{i+1}-u_i).$$

In the sequel we shall need the following.

Assumption A. Suppose the function $u = (u_0, \dots, u_{n+1})$ satisfies

(3) $$u_0 = u_n = 0 ,$$

(4) $$|u_i^{(2)}| \leqslant A \text{ (independent of } h) , \quad i = 1, \dots, n .$$

Suppose the coefficients b_i and c_i involved in (1) satisfy

(5) $$|b_i| \leqslant 2\beta, \beta \geqslant 0, c_i \leqslant \eta < 0 , \quad i = 1, \dots, n .$$

It is convenient to isolate the following simple fact as a

Lemma. Suppose the function u attains its maximum in some i_0 which is different from 0 and $n+1$. Then

(6) $$u_{i_0}^{(2)} \leqslant 0 \quad \text{and} \quad 2|u_{i_0}^{(1)}| \leqslant -hu_{i_0}^{(2)} .$$

Proof. Since $u_{i_0}^{(2)} = (u_{i_0+1} - u_{i_0}) + (u_{i_0-1} - u_{i_0})$, the first inequality of (6) follows from the fact that

$$u_i - u_{i_0} \leqslant 0 \quad \text{for all } i \neq 0 \quad \text{and} \quad n+1 .$$

For the same reason, the second term of (2) may be dropped. We get therefore the inequality

$$2u_{i_0}^{(1)} \leqslant -hu_{i_0}^{(2)}$$

Now we may replace in this inequality the function $(u_0, \dots, u_{i_0-1}, u_{i_0}, u_{i_0+1}, \dots, u_{n+1})$ by $(u_0, \dots, u_{i_0+1}, u_{i_0}, u_{i_0-1}, \dots, u_{n+1})$ (that is u_{i_0-1} and u_{i_0+1} only change places there). Thus we obtain

$$-2u_{i_0}^{(1)} \leqslant -hu_{i_0}^{(2)} .$$

This and the preceding inequality together imply the desired inequality of (6).

3. Now we are in a position to prove our

Theorem 1. Suppose Assumption A is satisfied. Suppose in addition the difference inequality (1) holds provided $u_i > 0$ and $i \neq 0$ and $n+1$. Then

(7) $$\max_{i=0,\dots,n+1} u_i \leqslant g(h)$$

where

(8) $$0 < g(h) = -\eta^{-1}(\beta hA + \varepsilon) .$$

Proof. Suppose to the contrary

(9) $$u_{i_0} = \max_{i=0,\dots,n+1} u_i > g(h) .$$

Because of (3), i_0 must be different from 0 and $n+1$. Thus u satisfies (1) with $i = i_0$. Using (6), (5), we check that

$$u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} + c_{i_0} u_{i_0} \leqslant -\tfrac{1}{2}|b_{i_0}| hu_{i_0}^{(2)} + c_{i_0} u_{i_0} \leqslant \beta hA + \eta u_{i_0} .$$

From (9), (8) it follows that

$$\beta h \Lambda + \eta u_{i_0} < -\varepsilon .$$

Thus we obtain the inequality

$$u_{i_0}^{(2)} + b_{i_0} u_{i_0}^{(1)} + c_{i_0} u_{i_0} < -\varepsilon .$$

Since, by (8) and (9), $u_{i_0} > 0$, u satisfies (1) with $i = i_0$. This contradits the above inequality.

4. Suppose $u = (u_0, \ldots, u_{n+1})$ satisfies the following inequality

$$u_i^{(2)} + b_i u_i^{(1)} + c_i u_i \leqslant \varepsilon; \tag{10}$$

Then $v_i = -u_i$ satisfies (1) and theorem 1 immediately implies the following

Theorem 2. Suppose Assumption A is satisfied. Suppose, in addition, the difference inequality (10) holds provided $u_i < 0$ and $i \neq 0$ and $n+1$. Then

$$\min_{i=0,\ldots,n+1} u_i \geqslant -g(h)$$

where $g(h)$ is defined by (8).

REFERENCES

[1] Z. Kowalski, A. Pliś, *Difference inequalities of the elliptic type*, Ann. Polon. Math., 26 (1972), 239—251.