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On the Approximation and Interpolation by Multiple Splines

1. INTRODUCTION

Let $\Delta = \{0 = x_0 < x_1 < \dots < x_N = 1\}$ be a partition of the interval $I = [0, 1]$. The function $S_\Delta \in C^{n-k}_{[0,1]}$ is called a spline of degree n and deficiency k with respect to the partition Δ if it is in each subinterval $I_i = [x_{i-1}, x_i]$ $i = (0, 1, \dots, N)$ a polynomial of degree at most n. We say that a spline S_Δ is simple if it is of deficiency 1 [1].

Let $\omega(t)$ be a modulus of continuity on I [2] and let H_ω denote a class of functions such that $\omega(f, t) \leqslant \omega(t)$.

V. L. Velikin [3] has proved that

$$(*)\qquad \sup_{f\in H_\omega} \|f - S_\Delta\| \leqslant \frac{3}{2}\omega\left(\frac{\|\Delta\|}{2}\right),$$

and if ω is concave

$$\sup_{f\in H_\omega} \|f - S_\Delta\| = \omega\left(\frac{\|\Delta\|}{2}\right),$$

where S_Δ is a spline of degree $2n-1$ and deficiency n, which interpolates to f at the points x_i, $i = 0, 1, \dots, N$ and such that $S_\Delta^{(k)}(x_i) = 0$ for $k = 1, 2, \dots, n-1$, $\|\Delta\| = \max_i (x_i - x_{i-1})$, $\omega(f, t) = \sup_{\substack{x, x+h \in I \\ 0<h\leqslant t}} |f(x+h) - f(x)|$, and $\|f\| = \sup_{x\in I} |f(x)|$. Moreover, the estimate $(*)$ is sharp.

In this paper theorems like this will be given for simple splines or multiple splines.

2. APPROXIMATION AND INTERPOLATION BY SIMPLE SPLINES

Let f be given on I and $\varphi \in C^{n-1}_{[0,1]}$. We divide each subinterval I_i into n parts by the points $\bar{x}_{ik}$, $k = 1, 2, \ldots, n-1$, $x_{i-1} = \bar{x}_{i0}$, $x_i = \bar{x}_{in}$. Let Δ' denote this subpartition. For each subinterval I_i a simple spline can be written in the form

$$(1) \qquad S_{\Delta'}(x) = \sum_{j=0}^{n-1} a_j (x-\bar{x}_{i0})^j + \sum_{k=0}^{N-1} \tau_k (x-\bar{x}_{ik})^n_+ , \qquad \text{where}$$

$$(x-t)^n_+ = \begin{cases} (x-t)^n & \text{for} \quad x \geqslant t \\ 0 & \text{for} \quad x < t \end{cases}$$

The conditions

$$(2) \qquad S^{(r)}_{\Delta'}(x_{i-1}) = \varphi^{(r)}(x_{i-1}) , \qquad S^{(r)}_{\Delta'}(x_i) = \varphi^{(r)}(x_i) , \qquad r = 0, 1, \ldots, n-1$$

imply that $a_j = \dfrac{\varphi^{(j)}(x_{i-1})}{j!}$ $j = (0, 1, \ldots, n-1)$. On substituting a_j into the remaining conditions, we obtain the n equations system with n unknowns τ_k, whose determinant is the Vandermonde determinant. Then the spline $S_{\Delta'}$ is unique if $S_{\Delta'}$ satisfies the system (2), where $\varphi \in C^{n-1}$ is given.

Lemma. Let $\Delta_1 = \{0 = x_0 < x_1 < \ldots < x_n = h\}$ be a partition of the interval $[0, h]$ and let H be a simple spline of degree n with respect to the partition Δ_1 satisfying the conditions: $H(h) = 1$, $H(0) = H^{(r)}(0) = H^{(r)}(h) = 0$, $r = 1, 2, \ldots, n-1$. Then the function H is increasing.

Proof. The function $H^{(n-1)}$ is a broken line and $H^{(n-1)}(0) = H^{(n-1)}(h) = 0$. We conclude from the definition of H that $H^{(n-1)}$ can change the sign at most $n-2$ times within the interval $(0, h)$. Therefore $H^{(n-2)}$ can have at most $n-2$ local extreme values in $(0, h)$. Then $H^{(n-2)}$ can have at most $n-3$ zeros at which it can change the sign. Repeating this reasoning $n-3$ times, we obtain that H' has exactly one extreme in $(0, h)$ and $H'(x) \neq 0$ for $x \in (0, h)$; but $H(h) = 1 > 0$, then $H'(x) > 0$ in $(0, h)$.

Remark 1. If G is a simple spline of degree n with respect to the partition Δ_1 satisfying the conditions: $G(0) = 1$, $G(h) = G^{(r)}(0) = G^{(r)}(h) = 0$, $r = 1, 2, \ldots, n-1$, then G is decreasing.

The proof is identical with the previous one.

Remark 2. The function $S(x) = G(x) + H(x)$ is the spline satisfying the conditions: $S(0) = S(h) = 1$, $S^{(r)}(0) = S^{(r)}(h) = 0$, $r = 1, 2, \ldots, n-1$. Then (1) implies that

$$(3) \qquad G(x) + H(x) \equiv 1$$

Remark 3. If Δ^* is the partition of the interval $[0, h]$ into n equal parts, then $G(x) = H(h-x)$ and $G\left(\frac{h}{2}\right) = H\left(\frac{h}{2}\right) = \frac{1}{2}$.

Let $S_{\Delta'}(f, x)$ denote a spline of degree n with respect to the partition Δ' satisfying the conditions:

$$S_{\Delta'}(f, x_i) = f(x_i),\ S_{\Delta'}^{(r)}(f, x_i) = 0,\ i = 0, 1, \ldots, N,\ r = 1, 2, \ldots, n-1. \tag{4}$$

Theorem 1. Let $\omega(t)$ be a modulus of continuity on I. Then

$$\sup_{f \in H_\omega} \sup_{\Delta', \|\Delta\| \leqslant h} \|f(x) - S_{\Delta'}(f, x)\| \leqslant \omega(h) \tag{5}$$

Proof. Let $f \in H_\omega$, $\|\Delta\| \leqslant h$, and $x \in I_{i+1}$. Then we can write the function $S_{\Delta'}(f, x)$ in the form $S_{\Delta'}(f, x) = f(x_i)G(x-x_i) + f(x_{i+1})H(x-x_i) = f(x_i) + [f(x_{i+1}) - f(x_i)]H(x-x_i)$. Because the function H is increasing therefore $f(x_i) \leqslant S_{\Delta'}(f, x) \leqslant f(x_{i+1})$ or $f(x_i) \geqslant S_{\Delta'}(f, x) \geqslant f(x_{i+1})$. Then $|f(x) - S_{\Delta'}(f, x)| \leqslant \max(|f(x) - f(x_i)|, |f(x) - f(x_{i+1})|) \leqslant \omega(f, x_{i+1} - x_i) \leqslant \omega(f, h)$. Since the interval I_{i+1} was chosen arbitrarily, the theorem is proved.

Corollary. Let f be a given function on I. Then

$$E_h^n(f) \leqslant \omega(f, nh), \quad n = 1, 2, \ldots, \tag{6}$$

where

$$E_h^n(f) = \sup_{\|\Delta\| \leqslant h} \inf_{S_\Delta} \|f - S_\Delta\|, \ \deg S \leqslant n. \tag{7}$$

Proof. Let Δ be a partition with $\|\Delta\| \leqslant h$. If N is divisible by n, $N = Kn$, then we take the spline S_Δ of Theorem 1 which interpolates to f at the points x_{kn}, $k = 0, 1, \ldots, K$ and for this spline $\|f - S_\Delta\| \leqslant \omega(f, nh)$. Otherwise, we can prolong the function f over the point $x = 1$ by setting $f^*(x) = f(x)$ for $x \leqslant 1$ and $f^*(x) = f(1)$ for $x > 1$ and choosing points $x_{N+1}, \ldots, x_{N+l}$ such that $N+l$ is divisible by n with $\|\Delta_1\| \leqslant h$ for the partition Δ_1 which we have just obtained. Now we can again apply Theorem 1. Since the partition Δ was chosen arbitrarily, the inequality (6) is proved.

Let Δ^* be a subpartition of the partition Δ, which we have obtained by dividing each interval I_k into n equal parts.

For this partition we have

Theorem 2. Let $\omega(t)$ be a modulus of continuity. Then

$$\sup_{f \in H_\omega} \|f(x) - S_{\Delta^*}(f, x)\| \leqslant \frac{3}{2}\omega\left(\frac{\|\Delta\|}{2}\right) \tag{8}$$

and if ω is concave

$$\sup_{f\in H_\omega}\|f(x)-S_{\Delta^*}(f,x)\| = \omega\left(\frac{\|\Delta\|}{2}\right). \tag{9}$$

The estimate in (8) is sharp.

The proof is nearly identical with the proof of Theorem 1 in [3]. Let $f\in H_\omega$ and $x\in I_i$. Set $\delta = x-x_{i-1}$, $x_i-x_{i-1} = h_i$. From remarks 2 and 3 we obtain

$$\begin{aligned}|f(x)-S_{\Delta^*}(f,x)| &= |[f(x)-f(x_{i-1})]G(\delta)+[f(x)-f(x_i)]G(h_i-\delta)|\\ &\leqslant \omega(\delta)G(\delta)+\omega(h_i-\delta)G(h_i-\delta)+\left[G(\delta)-\frac{1}{2}\right][\omega(h_i-\delta)-\omega(\delta)]\\ &= \frac{1}{2}[\omega(\delta)+\omega(h_i-\delta)] \leqslant \frac{3}{2}\omega\left(\frac{h_i}{2}\right)\leqslant \frac{3}{2}\omega\left(\frac{\|\Delta\|}{2}\right).\end{aligned}$$

If ω is concave, then

$$\omega(\delta)+\omega(h_i-\delta)\leqslant 2\omega\left(\frac{h_i}{2}\right).$$

Hence

$$\|f(x)-S_{\Delta^*}(f,x)\|\leqslant \omega\left(\frac{\|\Delta\|}{2}\right).$$

The equality in (9) is satisfied for the function

$$f_1^*(x) = \begin{cases}\omega(x-x_{i-1}) & \text{for} \quad x_{i-1}\leqslant x\leqslant x_{i-1}+\dfrac{h_i}{2}\\ \omega(x_i-x) & \text{for} \quad x_{i-1}+\dfrac{h_i}{2}\leqslant x\leqslant x_i\\ 0 & \text{for} \quad x\in I\setminus I_i\end{cases} \tag{10}$$

and i is chosen such that $h_i = \|\Delta\|$

Now we shall show that the estimate in (8) is sharp.

Given $\varepsilon > 0$ take ξ such that $0<\xi<\dfrac{\|\Delta\|}{4}$ and

$$H(\delta)<\frac{1}{2}+\varepsilon \quad \text{for} \quad \delta\leqslant\frac{\|\Delta\|}{2}+\xi. \tag{11}$$

Define the function f_2^* as follows

$$(12)\qquad f_2^*(x)=\begin{cases} 0 & \text{for} \quad x\in[0,x_{i-1}] \\ \frac{1}{\xi}(x-x_{i-1}) & \text{for} \quad x\in[x_{i-1},x_{i-1}+\xi] \\ 1 & \text{for} \quad x\in\left[x_{i-1}+\xi,\, x_{i-1}+\frac{h_i}{2}\right] \\ \frac{1}{\xi}\left(x-x_{i-1}-\frac{h_i}{2}\right)+1 & \text{for} \quad x\in\left[x_{i-1}+\frac{h_i}{2},\, x_{i-1}+\frac{h_i}{2}+\xi\right] \\ 2 & \text{for} \quad x\in\left[x_{i-1}+\frac{h_i}{2}+\xi,\, x_i-\xi\right] \\ 1-\frac{1}{\xi}(x-x_i+\xi) & \text{for} \quad x\in[x_i-\xi,x_i] \\ 1 & \text{for} \quad x\in[x_i,1] \end{cases}$$

where $h_i=\|\Delta\|$

Then $\omega\left(f_2^*,\frac{\|\Delta\|}{2}\right)=\omega\left(f_2^*,\frac{h_i}{2}\right)=1$ and at the point $x^*=x_{i-1}+\frac{h_i}{2}+\xi$ by (11) we obtain

$$\|f_2^*(x)-S_{\Delta^*}(f_2^*,x)\|\geqslant|f_2^*(x^*)-S_{\Delta^*}(f_2^*,x^*)|=|f_2^*(x^*)-H(x^*-x_{i-1})|$$
$$>2-\left(\frac{1}{2}+\varepsilon\right)=\left(\frac{3}{2}-\varepsilon\right)\omega\left(f_2^*,\frac{\|\Delta\|}{2}\right).$$

Remark. This theorem remains true if instead of the spline $S_{\Delta^*}(f,x)$ we take the function, which can be written in the form $f(x_{i-1})P(x-x_{i-1})+f(x_i)Q(x-x_{i-1})$ for $x\in I_i$, where the functions P and Q belong to the class C^n and satisfy the conditions: $P(\delta)+Q(\delta)=1$, $P(0)=1$, $P(h)=0$, $P^{(k)}(0)=P^{(k)}(h)=0$, $k=1,2,\dots,n$, $P\left(\frac{h}{2}\right)=\frac{1}{2}$, $0\leqslant P(\delta)\leqslant\frac{h}{2}$ for $0\leqslant\delta\leqslant\frac{h}{2}$, $\frac{1}{2}\leqslant P(\delta)\leqslant 1$ for $\frac{h}{2}\leqslant\delta\leqslant h$, and for each interval I_i we set $h=h_i$.

3. APPROXIMATION AND INTERPOLATION IN R^n

For convenience we introduce the following denotations: $m=(m_1,\dots,m_n)$, $k=(k_1,\dots,k_n)$, $x=(x_1,\dots,x_n)$, $x\leqslant y \overset{\text{df}}{\Leftrightarrow} x_i\leqslant y_i$ for $i=1,2,\dots,n$, $\Delta_j=\{0=\overset{0}{x}_j<\overset{1}{x}_j<\dots<\overset{N_j}{x}_j=1\}$, $\Delta=(\Delta_1,\dots,\Delta_n)$, $|p|=p_1+\dots+p_n$.

We say that a function S_Δ is a multiple spline of degree m and deficiency k with respect to the partition Δ, if S_Δ as a function of x_j, when the remaining variables are fixed is a spline of degree m_j and deficiency k_j with respect to the partition Δ_j. It follows from the definition that all the derivatives

$$(13)\qquad D^p S_\Delta = \frac{\partial S_\Delta^{|p|}}{\partial x_1^{p_1} \dots \partial x_n^{p_n}}, \quad 0 \leqslant p_j \leqslant m_j - k_j, \quad j = 1, 2, \dots, n$$

are continuous in I^n. We say that a spline S_Δ is simple with respect to the partition Δ, if $k = (1, 1, \dots, 1)$.

Let Δ' denote the subpartition of the partition Δ, which is obtained by dividing into m_j parts each subinterval of the partition Δ_j. Let φ be a given function of n variables on I^n with continuous derivatives (13).

Further we need the following.

Lemma 2. The simple spline $S_{\Delta'}$ satisfying the conditions:

$$(14)\quad D^p S_{\Delta'}(\overset{i}{x}) = D^p \varphi(\overset{i}{x}), \quad \overset{i}{x} = (\overset{i_1}{x}_1, \dots, \overset{i_n}{x}_n), \quad i_j = 0, 1, \dots, N_j,$$
$$j = 1, 2, \dots, n, \quad 0 \leqslant p_j \leqslant m_j - 1$$

is unique.

Proof. For $n = 1$ the uniqueness has already been proved. Suppose that the theorem is true for splines of $n-1$ variables. Then the functions $\frac{\partial^{p_n}}{\partial x^{p_n}} S_{\Delta'}(\bar{x}, \overset{i_n}{x}_n)$, where $\bar{x} = (x_1, \dots, x_{n-1})$ satisfy the assumption of the theorem for $k = n-1$ and the inductive assumption implies that they are unique. For any fixed $\bar{x}$ the function $S_{\Delta'}(\bar{x}, x_n)$ is a simple spline of the single variable x_n, which has given values of the first $m_n - 1$ derivatives at the points of the partition Δ_n, and this suffices for the uniqueness.

Let $S_{\Delta'}(f, x)$ denote as before a simple spline satisfying the conditions:

$$(15)\quad S_{\Delta'}(f, \overset{i}{x}) = f(\overset{i}{x}) \quad \text{and} \quad S^p S_{\Delta'}(f, \overset{i}{x}) = 0, \quad |p| \geqslant 1,$$
$$i_j = 0, 1, \dots, N_j, \quad j = 1, 2, \dots, n$$

and let $\omega(f, h)$ denote a continuity modulus of f

$$(16)\qquad \omega(f, h) = \sup_{\substack{x, x+t \in I^n \\ 0 < t \leqslant h}} |f(x+t) - f(x)|$$

Let $\omega(t)$ be a modulus of continuity on I^n and let H_ω denote a class of functions such that $\omega(f, t) \leqslant \omega(t)$.

Theorem 3. Let $\omega(t)$ be a modulus of continuity on I^n. Then

$$(17)\qquad \sup_{f \in H_\omega} \sup_{\Delta', (\Delta) \leqslant h} \|f(x) - S_{\Delta'}(f, x)\| \leqslant \omega(h),$$

where $(\Delta) = (\|\Delta_1\|, \dots, \|\Delta_n\|)$.

Proof. Let $I_{i_1}\times\ldots\times I_{i_n}=I^*$, $f\in H_\omega$, $(\Delta)\leqslant h$, $x\in I^*$, $m=\inf_{x\in I^*} f(x)$, and $M=\sup_{x\in I^*} f(x)$. It follows from Theorem 1 that for $k=1$

$$m\leqslant S_{\Delta'}(f,x)\leqslant M \tag{18}$$

Suppose that (18) is true for $k=n-1$. Then for each fixed $\bar{x}$ $m\leqslant S_{\Delta'}(f;x,x_n)\leqslant M$ and the function $S_{\Delta'}(f;x,x_n)$ is a simple spline satisfying the assumption of Theorem 1. Hence by the monotonicity of this function and in view of the arbitrariness of $\bar{x}$

$$m\leqslant S_{\Delta'}(f,x)\leqslant M$$

Further

$$|f(x)-S_{\Delta'}(f,x)|\leqslant \max(|f(x)-m|,|f(x)-M|)\leqslant\omega(h)$$

Since the interval I^* was chosen arbitrarily, the theorem is proved.

Corollary. Let f be a given function on I^n. Then

$$E_h^m(f)\leqslant\omega(f;m_1h_1,\ldots,m_nh_n)\,, \tag{19}$$

where

$$E_h^m(f)=\sup_{(\Delta)\leqslant h}\inf_{S_\Delta}\|f-S_\Delta\|,\ \deg S_\Delta\leqslant m\,. \tag{20}$$

Proof. Let Δ be a partition with $(\Delta)\leqslant h$. If N_k are divisible by m_k, $N_k=a_km_k$ for $k=1,2,\ldots,n$, then we take the spline S_Δ of Theorem 3 interpolating f at the points $\overset{i}{x}=(\beta_{i_1}m_1,\ldots,\beta_{i_n}m_n)$, $\beta_{i_k}=0,1,\ldots,a_k$, $k=1,2,\ldots,n$ and for this spline $\|f-S_\Delta\|\leqslant\omega(f;m_1h_1,\ldots,m_nh_n)$. Otherwise, we can prolong the function f over the interval I^n as follows. Define a sequence of functions and partitions: $f_0(x)=f(x)$; $f_1(x)=f_0(x)$ for $x\in I^n$, $f_1(x)=f_1(1,x_2,\ldots,x_n)$ for $x_1>1$, $x_i\leqslant 1$, $i=2,3,\ldots,n$. We choose points $\overset{N_1+1}{x_1},\ldots,\overset{N_1+l_1}{x_1}$ such that N_1+l_1 is divisible by m_1 with $\|\Delta_1''\|\leqslant h_1$ for the partition Δ_1'', which we have just obtained. $f_k(x)=f_{k-1}(x)$ for $x_i\leqslant\overset{N_i+l_i}{x_i}$, $i=1,2,\ldots,k-1$, $x_j\leqslant 1$, $j=k,\ldots,n$, $f_k(x)=$ $=f_k(x_1,\ldots,k_{k-1},1,x_{k+1},\ldots,x_n)$ for $x_i\leqslant\overset{N_i+l_i}{x_i}$, $i=1,2,\ldots,k-1$, $x_k>1$, $x_j\leqslant 1$, $j=k+1,\ldots,n$. We choose points $\overset{N_k+1}{x_k},\ldots,\overset{N_k+l_k}{x_k}$ such that N_k+l_k is divisible by m_k with $\|\Delta_k\|\leqslant h_k$ for the partition Δ_k which we have just obtained. Then the function $f_n(x)$ is defined on $[0,\overset{N_1+l_1}{x_1}]\times\ldots\times[0,\overset{N_n+l_n}{x_n}]$, $f_n(x)=f(x)$ for $x\in I^n$, and $\omega(f_n,t)=\omega(f,t)$.

Now we can again apply Theorem 3. Since the partition Δ was chosen arbitrarily, the theorem is proved.

Let Δ^* be a subpartition of the partition Δ, which we have obtained by dividing each subinterval of the partition Δ_i into m_i equal parts. For this partition we have

Theorem 4. Let $\omega(t)$ be a modulus of continuity. Then

$$\sup_{f\in H_\omega}\|f(x)-S_{\Delta^*}(f,x)\| \leqslant \left(2-\frac{1}{2^n}\right)\omega\left(\frac{(\Delta)}{2}\right) \tag{21}$$

$$\sup_{f\in H_\omega}\|f(x)-S_{\Delta^*}(f,x)\| \leqslant \frac{3}{2}\omega\left(\frac{\|\Delta_1\|}{2},0,\ldots,0\right)+\ldots+\omega\left(0,\ldots,0,\frac{\|\Delta_n\|}{2}\right) \tag{22}$$

and if ω is concave with respect of each variable

$$\sup_{f\in H_\omega}\|f(x)-S_{\Delta^*}(f,x)\| = \omega\left(\frac{(\Delta)}{2}\right). \tag{23}$$

The estimates in (21) and (22) are sharp.

Proof. Let $n=2$ and $x\in I_j\times I_k$. Define the functions

$$\begin{aligned} K_1(x) &= G_1(x_1)\cdot G_2(x_2)\\ K_2(x) &= H_1(x_1)\cdot G_2(x_2)\\ K_3(x) &= H_1(x_1)\cdot H_2(x_2)\\ K_4(x) &= G_1(x_1)\cdot H_2(x_2)\end{aligned} \tag{24}$$

where H_i and G_i are the functions of Lemma 1 defined for the partition Δ_i respectively in I_j and I_k.
It follows from Lemma 2, the definition of the partition Δ^*, and Remark 3 of Lemma 1 that

$$K_1(x)+K_2(x)+K_3(x)+K_4(x)=1 \tag{25}$$

Let $f\in H_\omega$. Set $x_i-\overset{j-1}{x_i}=\delta_i$, $\overset{j}{x_1}-\overset{j-1}{x_1}=a_j$, $\overset{k}{x_2}-\overset{k-1}{x_2}=\beta_k$, $f(\overset{j}{x_1},\overset{k}{x_2})=f_{j,k}$. We can write the function in the form

$$S_{\Delta^*}(f,x)=f_{j-1,k-1}G_1(\delta_1)G_2(\delta_2)+f_{j,k-1}G_1(a_j-\delta_1)G_2(\delta_2)+$$
$$+f_{j,k}G_1(a_j-\delta_1)G_2(\beta_k-\delta_2)+f_{j-1,k}G_1(\delta_1)G_2(\beta_k-\delta_2)$$

Hence from (25) and the proof of Theorem 2

$$|f(x)-S_{\Delta^*}(f,x)| = |[f(x)-f_{j-1,k-1}]G_1(\delta_1)G_2(\delta_2)+$$
$$+[f(x)-f_{j,k-1}]G_1(a_j-\delta_1)G_2(\delta_2)+[f(x)-f_{j,k}]G_1(a_j-\delta_1)G_2(\beta_k-\delta_2)+$$
$$+[f(x)-f_{j-1,k}]G_1(\delta_1)G_2(\beta_k-\delta_2)| \leqslant \omega(\delta_1,\delta_2)G_1(\delta_1)G_2(\delta_2)+$$
$$+\omega(a_j-\delta_1,\delta_2)G_1(a_j-\delta_1)G_2(\delta_2)+\omega(a_j-\delta_1,\beta_k-\delta_2)G_1(a_j-\delta_1)G_2(\beta_k-\delta_2)+$$
$$+\omega(\delta_1,\beta_k-\delta_2)G_1(\delta_1)G_2(\beta_k-\delta_2)\leqslant\frac{1}{2}[\omega(\delta_1,\delta_2)+\omega(a_j-\delta_1,\delta_2)]G_2(\delta_2)+$$

$$+\frac{1}{2}[\omega(\alpha_j-\delta_1, \beta_k-\delta_2)+\omega(\delta_1, \beta_k-\delta_2)]G_2(\beta_k-\delta_2)$$

$$\leqslant\frac{1}{4}[\omega(\delta_1, \delta_2)+\omega(\alpha_j-\delta_1, \delta_2)+\omega(\alpha_j-\delta_1, \beta_k-\delta_2)+\omega(\delta_1, \beta_k-\delta_2)]$$

$$\leqslant\left(2-\frac{1}{4}\right)\omega\left(\frac{\alpha_j}{2}, \frac{\beta_k}{2}\right)\leqslant\frac{7}{4}\omega\left(\frac{(\Delta)}{2}\right).$$

If $\omega(t)$ is concave with respect to each variable, then $\omega(\delta_1, \delta_2)+\omega(\alpha_j-\delta_1, \delta_2)+$ $+\omega(\alpha_j-\delta_1, \beta_k-\delta_2)+\omega(\delta_1, \beta_k-\delta_2)\leqslant 4\omega\left(\frac{\alpha_j}{2}, \frac{\beta_k}{2}\right)\leqslant 4\omega\left(\frac{(\Delta)}{2}\right)$ therefore in this case

$$\sup_{f\in H_\omega}\|f(x)-S_{\Delta^*}(f, x)\|\leqslant\omega\left(\frac{(\Delta)}{2}\right). \tag{26}$$

To prove the inequality (22), write the function $S_{\Delta^*}(f, x)$ in the form

$$S_{\Delta^*}(f; x_1, x_2)=S_{\Delta^*}(f; \overset{j-1}{x_1}, x_2)G_1(\delta_1)+S_{\Delta^*}(f; \overset{j}{x_1}, x_2)G_1(\alpha_j-\delta_1).$$

Denote

$$R_{\Delta^*}(x_1, x_2)=f(\overset{j-1}{x_1}, x_2)G_1(\delta_1)+f(\overset{j}{x_1}, x_2)G_1(\alpha_j-\delta_1)$$

Then the function $R_{\Delta^*}(x_1, x_2)-S_{\Delta^*}(f; x_1, x_2)=[f(\overset{j-1}{x_1}, x_2)-S_{\Delta^*}(f; \overset{j-1}{x_1}, x_2)]G_1(\delta_1)+$ $+[f(\overset{j}{x_1}, x_2)-S_{\Delta^*}(f; \overset{j}{x_1}, x_2)]G_1(\alpha_j-\delta_1)$ is monotone with respect to x_1 for each fixed x_2. Then from Theorem 2 we see that

$$|R_{\Delta^*}(x_1, x_2)-S_{\Delta^*}(f; x_1, x_2)|\leqslant\max\{|f(\overset{j-1}{x_1}, x_2)-S_{\Delta^*}(\overset{j-1}{x_1}, x_2)|,$$

$$|f(\overset{j}{x_1}, x_2)-S_{\Delta^*}(\overset{j}{x_1}, x_2)|\}\leqslant\frac{3}{2}\omega\left(0, \frac{\beta_k}{2}\right).$$

Further

$$|f(x)-S_{\Delta^*}(f, x)|\leqslant|f(x)-R_{\Delta^*}(x)|+|R_{\Delta^*}(x)-S_{\Delta^*}(f, x)|\leqslant\frac{3}{2}\omega\left(\frac{\alpha_j}{2}, 0\right)+\frac{3}{2}\omega\left(0, \frac{\beta_k}{2}\right)$$

whence we obtain (22).

Applying these methods, we can easily prove by induction the formulas (21), (22) and (26) for multisplines.

To complete the proof we shall give some suitable examples.

Let $\omega(t)$ be concave with respect to each variable, $f\in H_\omega$, and $I^*=I_{i_1}\times\ldots\times I_{i_n}$ such that $(\overset{i_k}{x_k}-\overset{i_k-1}{x_k})=\|\Delta_k\|$, $k=1, 2, \ldots, n$. Denote

$$\overset{i-1}{x}=(\overset{i_1-1}{x_1}, \ldots, \overset{i_n-1}{x_n}), \qquad \overset{i}{x}=(\overset{i_1}{x_1}, \ldots, \overset{i_n}{x_n})$$

Define the sequence of functions

$$g_0(x) = \begin{cases} 0 & \text{for} \quad x \leqslant \overset{i-1}{x} \\ \omega(x - \overset{i-1}{x}) & \text{for} \quad \overset{i-1}{x} \leqslant x \leqslant \frac{1}{2}(\overset{i-1}{x} + \overset{i}{x}) = \overset{i-1}{x} + \frac{(\Delta)}{2}. \end{cases}$$

Forming the symmetric reflection of g_{k-1} with respect to the plane $x_k = \overset{i_k-1}{x_k} + \frac{\|\Delta_k\|}{2}$, we prolong g_{k-1} over this plane. Let g_k denote this function. Then $g_n \in H_\omega$ and satisfies (26), so we have (23).

Given $\varepsilon > 0$ take $\xi = (\xi_1, \ldots, \xi_n)$ such that $0 < \xi_k < \frac{\|\Delta_k\|}{4}$, $k = 1, 2, \ldots, n$ and

$$H_k(\delta_k) < \tfrac{1}{2} + \varepsilon \quad \text{for} \quad \delta_k \leqslant \frac{\|\Delta_k\|}{2} + \xi_k, \tag{27}$$

where H_k is the function of Lemma 1 defined with respect to the partition Δ_k. Then the function $f_3^*(x) = \sum_{k=1}^{n} f_k(x_k)$, where $f_k(x_k) = f_2^*(x_k)$ is defined by (12) with respect to Δ_k

$$\omega\left(f_3^*; \frac{\|\Delta_1\|}{2}, 0, \ldots, 0\right) = \omega\left(f_3^*; 0, \ldots, 0, \frac{\|\Delta_n\|}{2}\right) = 1$$

and at the point $x^* = \overset{i-1}{x} + \frac{(\Delta)}{2} + \xi$ by (27) we obtain

$$\begin{aligned} \|f_3^*(x) - S_{\Delta^*}(f_3^*, x)\| &\geqslant |f_3^*(x^*) - S_{\Delta^*}(f_3^*, x^*)| \\ &= \left|f_3^*(x^*) - \sum_{k=1}^{n} H_k(x_k^* - \overset{i_k-1}{x_k})\right| > n\left(\frac{3}{2} - \varepsilon\right) \\ &= \left(\frac{3}{2} - \varepsilon\right)\left[\omega\left(f_3^*; \frac{\|\Delta_1\|}{2}, 0, \ldots, 0\right) + \ldots + \omega\left(f_3^*; 0, \ldots, 0, \frac{\|\Delta_n\|}{2}\right)\right]. \end{aligned}$$

Then the estimate in (22) is sharp.

Let the function f be defined by (12) for $x_{i-1} = 0$, $x_i = 1$, $h_i = 1$ and $f(x) = 1$ for $x \geqslant 1$ and let be given the family of sets $A_{jk} = \{x: 0 \leqslant x_j \leqslant x_k\}$, $j, k = 1, 2, \ldots, n$, $j \neq k$ and the function $f_n(x) = f(x_j)$ for $x \in \bigcap_{\substack{k=1 \\ k \neq j}}^{n} A_{jk}$, $j = 1, 2, \ldots, n$.

It follows from the definition of the function f_n that $f_n = \text{const}$ on the planes $x_j = a$ for $x \in \bigcap_{\substack{k=1 \\ k \neq j}}^{n} A_{jk}$, $j = 1, 2, \ldots, n$. From here we can easily notice that

$\omega(f_n; \frac{1}{2}, \ldots, \frac{1}{2}) = 1$, $f_n(\frac{1}{2}+\xi, \ldots, \frac{1}{2}+\xi) = 2$ and on each plane $x_j = 0$, $j = 1, 2, \ldots, n$, $f_n(x) = 0$.

Define also the family of sets $B_k = \{x\colon 0 \leqslant x_k \leqslant \overset{i_k-1}{x_k}\}$, $k = 1, 2, \ldots, n$. For the partition Δ we can now define the function

$$f_4^*(x) = \begin{cases} 0 \quad \text{for} \quad x \in (\bigcap_{k=1}^{n} B_k) \cap I^n \\ f_n[\|\Delta_1\|(x_1 - \overset{i_1-1}{x_1}), \ldots, \|\Delta_n\|(x_n - \overset{i_n-1}{x_n})] \quad \text{for} \quad x \geqslant \overset{i-1}{x}\,. \end{cases} \tag{28}$$

Then $\omega\left(f_4^*, \frac{(\Delta)}{2}\right) = 1$ and for $x \in I^*$

$$S_{\Delta^*}(f_4^*, x) = H_1(x_1 - \overset{i_1-1}{x_1}) \cdot \ldots \cdot H_n(x_n - \overset{i_n-1}{x_n})\,, \qquad S_{\Delta^*}\left(f_4^*, \overset{i-1}{x} + \frac{(\Delta)}{2}\right) = \frac{1}{2^n}\,.$$

Let $\varepsilon > 0$. From the definition of the function $S_{\Delta^*}(f_4^*, x)$ we see that there exists a point $\eta = [\overset{i_1-1}{x_1} + \|\Delta_1\|(\frac{1}{2}+\xi), \ldots, \overset{i_n-1}{x_n} + \|\Delta_n\|(\frac{1}{2}+\xi)]$ such that

$$S_{\Delta^*}(f_4^*, \eta) < \frac{1}{2^n} + \varepsilon\,. \tag{29}$$

Hence

$$\|f_4^*(x) - S_{\Delta^*}(f_4^*, x)\| \geqslant |f_4^*(\eta) - S_{\Delta^*}(f_4^*, \eta)| > 2 - \left(\frac{1}{2^n} + \varepsilon\right)$$
$$= \left[\left(2 - \frac{1}{2^n}\right) - \varepsilon\right] \omega\left(f_4^*, \frac{(\Delta)}{2}\right)$$

and the theorem is proved.

Remark. These theorems will be true if we take splines of degree $2r-1 = (2r_1-1, \ldots, 2r_n-1)$ and deficiency $r = (r_1, \ldots, r_n)$ instead of simple splines. The proofs are analogous.

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