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Note on an Iterative Method for Solving Operator Equations

1. The paper by W. Solak [2] contains a theorem on a method of solving the abstract equation:

$$x = T(x)\,, \tag{1}$$

where $T: R \to R$ (R is an abstract space endowed with a metric $d: R \times R \to G$, where G is a partially ordered set and non-increasing sequences of elements of G have limits; both G and R have been introduced by T. Ważewski [3]) with the aid of the iterative formula:

$$x_n = T_n(x_{n-1}, y_{n-1})\,, \quad T_n: R \times R \to R\,, \quad n = 1, 2, \ldots \tag{2}$$

It turns out that this theorem contains incorrect estimates of the distances $d(x_n, x)$ and $d(y_n, x)$, where x is the solution of the equation (1). The lemma from [2] p. 222 cannot be used to obtain the inequalities on p. 224. On the other hand, the result of [2] is very close to one obtained by M. Kwapisz [1].

In the present note we formulate a corrected version of the theorem from [2] and discuss connections between [1] and [2]

2. Suppose we are given two non-increasing sequences $\{a_n\}$ and $\{c_n\}$ of elements of G and $a_n \searrow 0$ and $c_n \searrow 0$.

We shall accept the following hypotheses:

(i) x is a unique solution of equation (1) on a set $D \subset R$,

(ii) operators $T_n: R \times R \to R$ satisfy the conditions:

$$d\big(T_n(x, x), T(x)\big) \leqslant a_n\,, \quad n = 1, 2, \ldots$$

(iii) sequence $\{y_n\}$, $y_n \in D$ is chosen to satisfy the condition:

$$d(x_n, y_n) \leqslant c_n\,, \quad n = 1, 2, \ldots$$

(iv) there exists a function $\varphi : [0, b]+[0, b]\to G$, $b = \operatorname{diam} D$, non-decreasing and continuous in its domain of definition and such that for each $y, z, s, t \in D$ and $n = 1, 2, \ldots$ the inequalities:

$$d\big(T_n(y, z), T_n(s, t)\big) \leqslant \varphi\big(d(y, s), d(z, t)\big)$$

hold.

(v) $\varphi(b, b)+a_1+c_1 \leqslant b$.

(vi) $\varphi(u, u) = u$ if $u = 0$.

Theorem: If hypotheses (i)—(vi) are fulfilled then for any $x_0, y_0 \in D$ the sequences $\{x_n\}$ and $\{y_n\}$ defined by (2) converge to x and the following inequalities hold:

$$d(x_n, x) \leqslant B_n(b), \quad n = 0, 1, \ldots,$$
$$d(y_n, x) \leqslant C_n(b), \quad n = 0, 1, \ldots,$$

where

$$B_n(b) = \varphi\big(B_{n-1}(b), C_{n-1}(b)\big)+a_n, \quad n = 1, 2, \ldots, \quad B_0(b) = b,$$
$$C_n(b) = B_n(b)+c_n, \quad n = 1, 2, \ldots, \quad C_0(b) = b.$$

Proof: If we replaced our assumption (ii) by the following:

$$d\big(T_n(x_n, x_n), T(x_n)\big) \leqslant a_n, \quad n = 1, 2, \ldots,$$

then we should get:

$$d\big(x_{n+1}, T(x_n)\big) \leqslant d\big(T_n(x_n, y_n), T(x_n)\big) \leqslant d\big(T_n(x_n, x_n), T(x_n)\big)+ \\ +\varphi\big(0, d(x_n, y_n)\big) \leqslant a_n+\varphi(0, c_n) = b_n.$$

By (iv) and (vi) the sequence $\{b_n\}$ tends to zero so the main assumption of the theorem 2 by M. Kwapisz [1] is fulfilled, from which our theorem follows in this case.

If (ii) is assumed then the proof runs the same way as that of the Kwapisz theorem.

Thus this corrected version of the result of [2] becomes a slight modification of theorem 2 from [1].

REFERENCES

[1] M. Kwapisz, *On the convergence of approximate iterations for abstract equations*, Ann. Polon. Math. 22 (1969), p. 73—87.

[2] W. Solak, *An iterative method for solving operator equations*, Ann. Polon. Math. 25 (1971), p. 221—225.

[3] T. Ważewski, *Sur un procédé de prouver la convergence des approximations successives sans utilisation des séries de comparaison*, Bull. Acad. Sci., sér. sci. math. astr. et phys., 8, No. 1 (1960), p. 45—52.