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On a Certain Boundary Problem for the Biharmonic Equation

1. In this paper we shall construct Green's function by the method of symmetric images and with its help solve the boundary problem with Riquier's conditions for the biharmonic equation in the domain

$$(1)\qquad D=\Big\{X(x_1,\dots,x_n)\in E_n : 0<x_1<\infty,\ -\infty<x_j<\infty,\ j=2,\dots,n-1,\ 0<x_n\cos\frac{\pi}{k}<x_1\sin\frac{\pi}{k}\Big\},$$

where $n\geqslant 3$ and $k>2$ are positive integers and E_n denotes the Euclidean n — space.

The analogous problem for $n=2$ and $k>2$ was solved in paper [3].

2. Let us consider the biharmonic equation

$$(2)\qquad \Delta^2 u(X)=0$$

and the domain (1).

We look for the function $u(X)$ of class C^4 in D, satisfying in D the equation (2) and the boundary conditions

$$(3)\qquad \begin{cases} u(X)=f_1(X)=F_1(X') & \text{for}\quad X\in l_0\\ u(X)=f_2(X)=F_2(X') & \text{for}\quad X\in l_1\end{cases}$$

$$(4)\qquad \begin{cases} \Delta u(X)=f_3(X)=F_3(X') & \text{for}\quad X\in l_0\\ \Delta u(X)=f_4(X)=F_4(X') & \text{for}\quad X\in l_1\end{cases}$$

where l_0 denotes the hyperplane $x_n=0$ and l_i-the hyperplane

$$x_n\cos i\frac{\pi}{k}=x_1\sin i\frac{\pi}{k}\quad \text{for}\quad i=1,2,\dots,2k.$$

Let $X_1(x_{11}, \dots, x_{1n}) = X(x_1, \dots, x_n)$ be an arbitrary interior point of D and let $X_{i+1}(x_{i+1,1}, \dots, x_{i+1,n})$ denote a symmetric image of the point $X_i(x_{i1}, \dots, x_{in})$ with respect to the hyperplane l_i. We obtain the following connections between the coordinates $x_{i1}, \dots, x_{in}$ of the points X_i and the coordinates $x_1, \dots, x_n$ of the point X

$$(5)\qquad \begin{cases} x_{i1} = x_1 \cos i\dfrac{\pi}{k} + x_n \sin i\dfrac{\pi}{k} & \\ x_{i2} = x_2 & \\ \dots\dots\dots\dots\dots & \text{for } i = 2, 4, \dots, 2k \\ x_{i,n-1} = x_{n-1} & \\ x_{in} = x_1 \sin i\dfrac{\pi}{k} - x_n \cos i\dfrac{\pi}{k} & \\ & \\ x_{i1} = x_1 \cos(i-1)\dfrac{\pi}{k} - x_n \sin(i-1)\dfrac{\pi}{k} & \\ x_{i2} = x_2 & \\ \dots\dots\dots\dots\dots\dots\dots & \text{for } i = 1, 3, \dots, 2k-1 \\ x_{i,n-1} = x_{n-1} & \\ x_{in} = x_1 \sin(i-1)\dfrac{\pi}{k} + x_n \cos(i-1)\dfrac{\pi}{k}\,. & \end{cases}$$

Let $Y(y_1, \dots, y_n)$ be an arbitrary point of a set $Z = D \cup l_0 \cup l_1$,

$$\varrho_i = |X_i Y| \quad \text{for} \quad i = 1, 2, \dots, 2k$$

and

$$(6)\qquad \begin{cases} r_i = \varrho_i|_{Y \in l_0} = [(y_1 - x_1)^2 + \dots + (y_{n-1} - x_{i,n-1})^2 + x_{in}{}^2]^{\frac{1}{2}} \\ R_i = \varrho_i|_{Y \in l_1} = [(y_1 - x_{i1})^2 + \dots + (y_{n-1} - x_{i,n-1})^2 + (my_1 - x_{in})^2]^{\frac{1}{2}} \end{cases}$$

where $m = \operatorname{tg}\dfrac{\pi}{k}$.

Definition 1. The function $G(X, Y)$ is said to be Green's function for the biharmonic equation, the domain D and the boundary conditions (3) and (4) when it has the following properties:

1. $G(X, Y) = V(X, Y) - H(X, Y)$, where $V(X, Y)$ is the fundamental solution of the equation (2) and $H(X, Y)$ is biharmonic as a function of Y for $Y \in D$ and a fixed $X \in D$ and $H(X, Y)$ is of class C^3 for $Y \in Z$, $X \in D$.
2. $G(X, Y) = \Delta_Y G(X, Y) = 0$ for $Y \in \partial D$
3. $\lim \Delta_Y G(X, Y) = 0$, when $|OY| \to \infty$, for an arbitrary fixed $X \in D$.

Theorem 1. Green's function for the equation (2) and the domain D with the boundary conditions

$$G(X, Y) = \Delta_Y G(X, Y) = 0 \quad \text{for } Y \in l_0 \cup l_1 \tag{7}$$

has the form

$$G(X, Y) = \sum_{i=1}^{2k} (-1)_n^{i+1} \ln \varrho_i \quad \text{for } n = 4 \tag{8}$$

$$G(X, Y) = \sum_{i=1}^{2k} (-1)^{i+1} \varrho_i^{4-n} \quad \text{for } n \geqslant 3, n \neq 4 \tag{9}$$

Proof. We shall prove that the conditions 1—3 of Definition 1 are satisfied.

The condition 1 is immediately evident.

2. If $Y \in l_0$, then $r_i = r_{2k-i+1}$ for $i = 1, \ldots, k$. The terms of both the sums (8) and (9) for the indices i and $2k-i+1$ differ only in sign and in that case $G(X, Y) = 0$. Similarly if $Y \in l_1$, $R_2 = R_1$ and $R_i = R_{2k-i+3}$ for $i = 3, \ldots, k+1$ and the proof of the first of the conditions (7) is analogous. In the same way we can prove the second condition (7).

3. Let $\varrho = |OY|$. By the triangle inequality we have

$$\varrho - |OX| \leqslant \varrho_i \leqslant \varrho + |OX| \quad \text{for } i = 1, 2, \ldots, 2k, \tag{10}$$

since $|OX_i| = |OX|$ for every X_i.

We have from (8) and (10) for $n = 4$

$$|\Delta_Y G(X, Y)| \leqslant 2 \sum_{i=1}^{2k} \varrho_i^{-2} \leqslant 2 \cdot 2k (\varrho - |OX|)^{-2},$$

so $\lim\limits_{\varrho \to \infty} \Delta_Y G(X, Y) = 0$ for every fixed $X \in D$.

Similarly we have from (9) and (10) for $n \geqslant 3, n \neq 4$

$$|\Delta_Y G(X, Y)| \leqslant 2|n-4| \cdot 2k (\varrho - |OX|)^{2-n},$$

so the conditions 3 is also satisfied.

Theorem 2. The functions $G(X, Y)$ defined by (8) and (9) are symmetric functions of points X and Y for $X, Y \in D$.

The **proof** is the same in both cases.

Let $Y_1 = Y$ be an arbitrary fixed point in the interior D and let Y_{i+1} be a symmetric image of Y_i with respect to l_i.

Let $\bar{\varrho}_i = |Y_i X|$, $\omega_i = (l_0, \overline{OX_i})$, $\bar{\omega}_i = (l_0, \overline{OY_i})$ for $i = 1, 2, \ldots, 2k$. We have

$$\omega_i = \begin{cases} i\dfrac{\pi}{k} - \omega_1 & \text{for } i = 2, 4, \ldots, 2k \\ (i-1)\dfrac{\pi}{k} + \omega_1 & \text{for } i = 3, 5, \ldots, 2k-1 \end{cases}$$

$$\tag{11} \bar{\omega}_i = \begin{cases} i\dfrac{\pi}{k} - \bar{\omega}_1 & \text{for } i = 2, 4, \ldots, 2k \\ (i-1)\dfrac{\pi}{k} + \bar{\omega}_1 & \text{for } i = 3, 5, \ldots, 2k-1 . \end{cases}$$

Then

$$\varrho_i^2 = |\overline{OY} - \overline{OX_i}|^2 = |\overline{OY}|^2 + |\overline{OX_i}|^2 - 2|\overline{OY}| \cdot |\overline{OX_i}| \cos\varphi_i$$

$$\bar{\varrho}_i^2 = |\overline{OX} - \overline{OY_i}|^2 = |\overline{OX}|^2 + |\overline{OY_i}|^2 - 2|\overline{OX}| \cdot |\overline{OY_i}| \cos\bar{\varphi}_i ,$$

where $\varphi_i = \omega_i - \bar{\omega}_1$, $\bar{\varphi}_i = \bar{\omega}_i - \omega_1$. By (11) and equalities $|\overline{OX_i}| = |\overline{OX}|$, $|\overline{OY_i}| = |\overline{OY}|$ for $i = 1, 2, \ldots, 2k$ we obtain

$$\varrho_i = \bar{\varrho}_i \qquad \text{for } i = 2, 4, \ldots, 2k$$

$$\varrho_i = \bar{\varrho}_{2k-i+2} \qquad \text{for } i = 3, 5, \ldots, 2k-1$$

and obviously $\varrho_1 = \bar{\varrho}_1$. Therefore theorem 2 is proved.

4. We can show analogically as in [1] p. 200—202 that if the domain D is bounded, then the solution of the equation (2) with the boundary conditions

$$u(X) = \varphi(X) , \quad \Delta u(X) = \psi(X) \quad \text{for } X \in \partial D ,$$

if it exists, has the form

$$\tag{12} u(X) = C_n \int_{\partial D} \left[u(Y) \frac{d\Delta_Y G(X, Y)}{d\nu} + \Delta u(Y) \frac{dG(X, Y)}{d\nu} \right] d\sigma_Y ,$$

where $C_n = -\dfrac{1}{2(n-2)(n-4)\theta_n}$ for $n \geqslant 3$, $n \neq 4$; $C_4 = \dfrac{1}{4\theta_4}$, θ_n denotes a measure of surface of the unit sphere, ν is the inward normal.

We shall prove that by certain assumptions about the functions $F_1, \ldots, F_4$ the solution of the problem (2), (3), (4) in the unbounded domain D defined by (1) has also the form (12).

Let $E_{n-1}^{+} = \{Y(y_1, \ldots, y_{n-1}) \colon 0 < y_1 < \infty, \ -\infty < y_j < \infty, \quad j = 2, \ldots, n-1\}$.

In view of the boundary conditions (3) and (4), the formulas (8) and (9) and the symmetry of the points X_i with respect to l_0 and l_1 we obtain for $n \geqslant 3$

$$u(X) = A_1 \sum_{i=1}^{k} (-1)^{i+1} H_i^{(1)}(X) + A_2 \sum_{i=2}^{k+1} (-1)^i H_i^{(2)}(X) + \tag{13}$$

$$+ A_3 \sum_{i=1}^{k} (-1)^i H_i^{(3)}(X) + A_4 \sum_{i=2}^{k+1} (-1)^{i+1} H_i^{(4)}(X) ,$$

where $A_1 = A_2 = \dfrac{2}{\theta_n}$, $A_3 = A_4 = \dfrac{1}{(n-2)\theta_n}$,

$$H_i^{(1)}(X) = \int\limits_{E_{n-1}^+} F_1(Y) x_{in} r_i^{-n} dY$$

$$H_i^{(2)}(X) = \int\limits_{E_{n-1}^+} F_2(Y)(x_{in} - m x_{i1}) R_i^{-n} dY$$

$$H_i^{(3)}(X) = \int\limits_{E_{n-1}^+} F_3(Y) x_{in} r_i^{2-n} dY$$

$$H_i^{(4)}(X) = \int\limits_{E_{n-1}^+} F_4(Y)(x_{in} - m x_{i1}) R_i^{2-n} dY ,$$

$x_{i1}, \ldots, x_{in}$ are defined by (5) and r_i, R_i by (6).

5. Let $r, \omega_1, \ldots, \omega_{n-1}$ denote the spherical coordinates of the point $X \in E_n$. Then

$$x_j = r\varphi_j(\omega) \quad \text{for } j = 1, \ldots, n \tag{14}$$

and the Jacobian of the transformation (14) has the form

$$J_n(r, \omega) = r^{n-1} K_n(\omega) ,$$

where $\varphi_j(\omega)$ and $K_n(\omega)$ are certain functions of a point ω defined in the set $H = \left\{\omega(\omega_1, \ldots, \omega_{n-1}) \colon 0 < \omega_1 < 2\pi, -\dfrac{\pi}{2} < \omega_j < \dfrac{\pi}{2}, j = 2, \ldots, n-1\right\}$ ([2] p. 29—31).

Let us consider the closed rectangle $P \subset D$ defined by inequalities

$$P = \{X : 0 < a_j \leqslant x_j \leqslant A_j, j = 1, \ldots, n-1, 0 < b \leqslant x_n \leqslant B < m a_1\} ,$$

where a_j, A_j, b, B are arbitrary positive numbers.

Let

$$\bar{F}_j(Y) = \begin{cases} F_j(Y) & \text{for } Y \in E_{n-1}^+ \\ 0 & \text{for } Y \in E_{n-1} - E_{n-1}^+ \end{cases} \qquad j = 1, \ldots, 4 .$$

Introducing the notation $D^{a}f = \frac{\partial^{|a|}f}{\partial x_1^{a_1} \dots \partial x_n^{a_n}}$, where $a = (a_1, \dots, a_n)$, $|a| = a_1 + \dots + a_n$, a_j are positive integers, we can prove

Lemma 1. If the function $F_1(Y)$ is defined and continuous in E_{n-1}^{+} and the integral $\int\limits_{H} K_{n-1}(\omega)\,d\omega \int\limits_{A}^{\infty} |\overline{F}_1(r\varphi_1(\omega), \dots, r\varphi_{n-1}(\omega))| r^{-2}dr$ is convergent for every $A > 0$, then the integral $H_1^{(1)}(X)$ is a function of class C^4 in D and $D^{a}H_1^{(1)}(X) = \int\limits_{E_{n-1}^{+}} F_1(Y) D^{a}(x_n r_1^{-n})\,dY$, where $a_j = 0, 1, 2, 3, 4$ $(j = 1, \dots, n)$, $1 \leqslant |a| \leqslant 4$.

Proof. Let $K_R = \{Y \in E_{n-1}\colon |OY| \leqslant R\}$

$$\text{and} \quad H_R(X) = \int\limits_{E_{n-1} - K_R} \overline{F}_1(Y) x_n r_1^{-n}\,dY\,.$$

Let $X \in P$. By the triangle inequality we obtain

$$\tfrac{1}{2}|OY| \leqslant |XY| \leqslant 2|OY|$$

for a sufficiently large $|OY|$. Then

$$|H_R(X)| \leqslant 2^n B \int\limits_{E_{n-1} - K_R} |\overline{F}_1(Y)| \cdot |OY|^{-n}\,dY\,.$$

Applying the transformation (14) we have

$$|H_R(X)| \leqslant 2^n B \int\limits_{H} K_{n-1}(\omega)\,d\omega \int\limits_{R}^{\infty} |\overline{F}_1(r\varphi_1, \dots, r\varphi_{n-1})| r^{-n} \cdot r^{n-2}dr\,,$$

so, for arbitrary positive ε, we obtain

$$|H_R(X)| < \varepsilon$$

for $X \in P$ and $R > R_0(\varepsilon, P)$.

It follows by the continuity of the function $F_1(Y)$ and from the condition $x_n \geqslant b$ that the integral $H_1^{(1)}(X)$ exists also inside the ball K_R. Then $H_1^{(1)}(X)$ is uniformly convergent in the set P.

Let $I_{|a|}(X) = \int\limits_{E_{n-1}^{+}} F_1(Y) D^{a}(x_n r_1^{-n})\,dY$.

The kernels of the integrals $I_1(X)$ and $I_2(X)$ are linear combinations of the functions r_1^{-n}, $x_n r_1^{-n-2} P_1$ and $r^{-n-2}P_1$, $x_n r_1^{-n-4}(P_1)^2$ respectively, where P_1 is a certain homogeneous polynomial of the variables $y_1 - x_1, \dots, y_{n-1} - x_{n-1}, x_n$. We have inequalities

$$|P_1| \leqslant Cr_1\,, \qquad |x_n| \leqslant r_1\,,$$

where C is a certain positive constant.

We have the following estimations

$$\left|\int_{E_{n-1}-K_R} \bar{F}_1(Y)\, x_n r_1^{-n-2} P_1\, dY\right| \leqslant C \int_{E_{n-1}-K_R} |\bar{F}_1(Y)|\, r_1^{-n}\, dY$$

$$\left|\int_{E_{n-1}-K_R} \bar{F}_1(Y)\, r_1^{-n-2} P_1\, dY\right| \leqslant \frac{C}{b} \int_{E_{n-1}-K_R} |\bar{F}_1(Y)|\, r_1^{-n}\, dY$$

$$\left|\int_{E_{n-1}-K_R} \bar{F}_1(Y)\, x_n r_1^{-n-4} (P_1)^2\, dY\right| \leqslant \frac{C^2}{b} \int_{E_{n-1}-K_R} |\bar{F}_1(Y)|\, r_1^{-n}\, dY\,.$$

We can show in the same manner as in the first part of this proof that these integrals are arbitrarily small for a sufficiently large R.

We can prove in the same way that the integrals $I_3(X)$ and $I_4(X)$ are also uniformly convergent in P.

Thus the integral $H_1^{(1)}(X) \in C^4$ in D and its derivatives up to the order four may be found by differentiation under the sign of the integral.

Lemma 2. If the function $F_3(Y)$ is defined and continuous in E_{n-1}^+ and the integral $\int_H K_{n-1}(\omega)\, d\omega \int_A^\infty |\bar{F}_3(r\varphi_1, \ldots, r\varphi_{n-1})|\, dr$ is convergent for every $A > 0$, then the integral $H_1^{(3)}(X)$ is a function of class C^4 in D and $D^\alpha H_1^{(3)}(X) = \int_{E_{n-1}^+} F_3(Y)\, D^\alpha(x_n r_1^{2-n})\, dY$.

The proof of this Lemma is similar to that of Lemma 1.

It follows from Lemmas 1 and 2.

Lemma 3. If the functions $F_1(Y)$ and $F_2(Y)$ satisfy the assumptions of Lemma 1 and the functions $F_3(Y)$ and $F_4(Y)$ satisfy the assumptions of Lemma 2, then the function $u(X)$ defined by (13) is a biharmonic function in D.

Proof. We can show analogically as in the proof of Lemma 1 that the integrals $H_i^{(1)}(X)$ and $H_i^{(3)}(X)$ $(i = 2, 3, \ldots, k)$ are functions of class C^4 in sets $\Omega_i = \{X \in E_n : -\infty < x_j < \infty,\ j = 1, \ldots, n-1, x_{in} > 0\}$, and the integrals $H_i^{(2)}(X)$ and $H_i^{(4)}(X)$ $(i = 2, 3, \ldots, k+1)$ are functions of class C^4 in sets $\tilde{\Omega}_i = \{X \in E_n : -\infty < x_j < \infty,\ j = 1, \ldots, n-1, x_{in} - m x_{i1} > 0\}$. By equality

$$D = \Big(\bigcap_{i=1}^{k} \Omega_i\Big) \cap \Big(\bigcap_{i=2}^{k+1} \tilde{\Omega}_i\Big)$$

the function $u(X)$ is of class C^4 in D. Differentiating under the sign of the integral and using the symmetry of Green's function we obtain from (12)

$$\Delta u(X) = C_n \int_{l_0 \cup l_1} \Delta u(Y)\, \frac{d\Delta_X G(X, Y)}{d\nu}\, d\sigma_Y\,, \tag{15}$$

so $\Delta^2 u(X) = 0$.

6. Now we shall prove the boundary conditions (3) and (4). We have first

Lemma 4. For every point $Q(q_1, \ldots, q_n) \in E_n(q_n \geqslant 0)$ and for an arbitrary number $a > 0$ the integral

$$J(Q) = \int\limits_{|QY| \geqslant a} |QY|^{-n} dY ,$$

where $Y = Y(y_1, \ldots, y_{n-1}, 0)$, is convergent.

Proof. In view of the inequality

$$|Q'Y| \geqslant |QY| - |QQ'| \geqslant a - q_n \qquad \big(Q' = Q'(q_1, \ldots, q_{n-1}, 0)\big)$$

and applying the transformation $y_j - q_j = q_n r \varphi_j(\omega)$ $(j = 1, \ldots, n-1)$, where $r, \omega_1, \ldots, \omega_{n-2}$ are the spherical coordinates, we obtain for $q_n > 0$

$$|J(Q)| \leqslant \int\limits_{|Q'Y| \geqslant a - q_n} |QY|^{-n} dy_1 \ldots dy_{n-1} = \frac{1}{q_n}\theta_{n-1} \int\limits_{a-q_n}^{\infty} r^{n-2}(r^2+1)^{-\frac{n}{2}} dr$$

$$\leqslant \frac{1}{q_n}\theta_{n-1} \int\limits_{0}^{\infty} r^{n-2}(r^2+1)^{-\frac{n}{2}} dr = \frac{1}{q_n}\theta_{n-1} \int\limits_{0}^{\infty} (t^2+1)^{-\frac{n}{2}} dt .$$

In virtue of the formula for $I_n = \int (t^2+1)^{-\frac{n}{2}} dt \, (n \geqslant 3)$

$$I_n = \frac{1}{n-2} t(t^2+1)^{-\frac{n}{2}+1} + \frac{n-3}{n-2} I_{n-2} \tag{16}$$

we obtain the result.

The proof of the convergence of the integral $J(Q)$ for $q_n = 0$ is immediate.

Lemma 5. With the above notations we have

$$I = \int\limits_{E_{n-1}} x_n[(y_1 - x_1)^2 + \ldots + (y_{n-1} - x_{n-1})^2 + x_n^2]^{-\frac{n}{2}} dy_1 \ldots dy_{n-1} = \tfrac{1}{2}\theta_n .$$

Proof. Using the transformation $y_j - x_j = x_n r \varphi_j(\omega)$ $(j = 1, \ldots, n-1)$ we obtain

$$I = \theta_{n-1} I_1$$

where

$$\theta_{n-1} = \begin{cases} \dfrac{2(2\pi)^{\frac{n-2}{2}}}{(n-3)!!} & \text{for even } n \ (n \geqslant 4) \\[2ex] \dfrac{(2\pi)^{\frac{n-1}{2}}}{(n-3)!!} & \text{for odd } n \ (n > 3) \end{cases}$$

$$I_1 = \int\limits_{0}^{\infty} r^{n-2}(r^2+1)^{-\frac{n}{2}} dr$$

By (16) we have the result.

Now let $X(x_1, \ldots, x_n) \to X_0(x_{01}, \ldots, x_{0,n-1}, 0)$, $x_n > 0$.

Lemma 6. If the function $F_1(Y)$ is defined and continuous in E^+_{n-1} and the integral $\int\limits_{|QY| \geqslant a} |F_1(Y)| \cdot |QY|^{-n} dY$ is convergent for every $a > 0$ and $Q \in E_n$, then

$$\lim_{X \to X_0} A_1 H_1^{(1)}(X) = F_1(X_0'), \tag{17}$$

where $A_1 = \dfrac{2}{\theta_n}$.

Proof. Let $K_\delta = \{Y \in E_{n-1}\colon |X_0' Y| \leqslant \delta\}$. We can write the integral $H_1^{(1)}(X)$ in the form

$$H_1^{(1)}(X) = I_1(X) + I_2(X) + I_3(X)$$

where

$$\begin{cases} I_1(X) = \displaystyle\int\limits_{E_{n-1}} F_1(X_0') x_n r_1^{-n} dY \\ I_2(X) = \displaystyle\int\limits_{K_\delta} [\overline{F}_1(Y) - F_1(X_0')] x_n r_1^{-n} dY \\ I_3(X) = \displaystyle\int\limits_{E_{n-1} - K_\delta} [\overline{F}_1(Y) - F_1(X_0')] x_n r_1^{-n} dY \,. \end{cases} \tag{18}$$

By Lemma 5

$$I_1(X) = F_1(X_0') \cdot \tfrac{1}{2}\theta_n \tag{19}$$

Let ε be an arbitrary positive number. If δ is sufficiently small, then it follows from the continuity of the function $F_1(Y)$ in the point X_0' that

$$|I_2(X)| \leqslant \frac{\varepsilon}{2} \cdot \frac{1}{2} \theta_n \,. \tag{20}$$

For the integral $I_3(X)$ we have

$$|I_3(X)| \leqslant x_n \Big[\int\limits_{E_{n-1} - K_\delta} |\overline{F}_1(Y)| r_1^{-n} dY + |F_1(X_0')| \int\limits_{E_{n-1} - K_\delta} r_1^{-n} dY \Big]$$

Let $K_{\frac{\delta}{2}} = \left\{ X \in E_n\colon |XX_0| \leqslant \dfrac{\delta}{2} \right\}$ and let $X \in K_{\frac{\delta}{2}} \cap D$. By the triangle inequality

$$r_1 \geqslant |YX_0'| - |XX_0'| \geqslant \frac{\delta}{2} \,,$$

thus

$$|I_3(X)| \leqslant x_n \int\limits_{|XY| \geqslant \frac{\delta}{2}} |\overline{F}_1(Y)| \cdot |XY|^{-n} dY + x_n |F_1(X_0')| \int\limits_{|XY| \geqslant \frac{\delta}{2}} |XY|^{-n} dY \,.$$

The first of the integrals above is convergent by the assumption, the second by Lemma 4. We have in this case

$$\lim_{X\to X_0} I_3(X) = 0\,. \tag{21}$$

Therefore the connections (18)—(21) imply (17).

Lemma 7. If

1° The functions $F_1(Y), \dots, F_4(Y)$ are defined and continuous in E^+_{n-1}

2° The integrals $\int\limits_{|QY|\geqslant a} |\overline{F}_j(Y)|\cdot|QY|^{-n}dY$ $(j = 1, 2)$ are convergent for every $a > 0$ and $Q \in E_n$

3° So are the integrals $\int\limits_{|QY|\geqslant a} |\overline{F}_j(Y)|\cdot|QY|^{2-n}dY$ $(j = 3, 4)$

4° $\lim\limits_{Q\to Q_0} q_n \int\limits_{E^+_{n-1}} F_j(Y)\,|QY|^{2-n}dY = 0$ for $j = 3, 4$, where $Q_0 = Q_0\,(q_{01}, \dots, q_{0,n-1}, 0)$,

then the function $u(X)$ defined by (13) satisfies the condition

$$\lim_{X\to X_0} u(X) = F_1(X_0')\,. \tag{22}$$

Proof. Let $K_\delta = \{X \in E_n : |XX_0| \leqslant \delta\}$, where δ is a positive number and let $X \in K_\delta \cap D$. It follows from (5) that

$$\left\{\begin{array}{ll} \lim\limits_{X\to X_0} x_{i1} = \lim\limits_{X\to X_0} x_{i+1,1} & \text{for } i = 2, 4, \dots, 2k-2 \\[2ex] \lim\limits_{X\to X_0} x_{in} = \lim\limits_{X\to X_0} x_{i+1,n} & \text{for } i = 2, 4, \dots, 2k-2 \\[2ex] \lim\limits_{X\to X_0} x_{i2} = x_{02}, \dots, \lim\limits_{X\to X_0} x_{i,n-1} = x_{0,n-1} & \text{for } i = 1, 2, \dots, 2k\,. \end{array}\right. \tag{23}$$

We shall rewrite the formula (13) as

$$u(X) = \tfrac{1}{2}\sum_{i=1}^{2k}[A_1(-1)^{i+1}H_i^{(1)}(X) + A_2(-1)^i H_i^{(2)}(X) + \\ + A_3(-1)^i H_i^{(3)}(X) + A_4(-1)^{i+1}H_i^{(4)}(X)]\,.$$

The functions under the sign of the integral in $H_i^{(1)}(X)$ and $H_i^{(3)}(X)$ have absolutely integrable majorants for $i = 2, 3, \dots, 2k-1$ by the assumptions 2 and 3 respectively, because x_{in} are bounded and the distances r_i are bounded below by a certain positive number when δ is sufficiently small. Therefore the limit may be approached under the integral sign. In virtue of (23) we have

$$\lim_{X\to X_0}\sum_{i=2}^{2k-1}(-1)^{i+1}H_i^{(1)}(X) = \lim_{X\to X_0}\sum_{i=2}^{2k-1}(-1)^i H_i^{(3)}(X) = 0 \tag{24}$$

and by the assumption 4

$$\lim_{X\to X_0} H_1^{(3)}(X) = \lim_{X\to X_0} H_{2k}^{(3)}(X) = 0\,. \tag{25}$$

Similarly the functions under the sign of the integral in $H_i^{(2)}$ and $H_i^{(4)}$ have absolutely integrable majorants for $i = 1, \dots, 2k$, because all R_i are greater than a certain positive number and we have in view of (23)

$$\lim_{X\to X_0} \sum_{i=2}^{2k-1} (-1)^i H_i^{(2)}(X) = \lim_{X\to X_0} \sum_{i=2}^{2k-1} (-1)^{i+1} H_i^{(4)}(X) = 0 \tag{26}$$

and from (5) immediately

$$\lim_{X\to X_0} [-H_1^{(2)}(X) + H_{2k}^{(2)}(X)] = \lim_{X\to X_0} [H_1^{(4)}(X) - H_{2k}^{(4)}(X)] = 0\,. \tag{27}$$

Besides

$$H_1^{(1)}(X) - H_{2k}^{(1)}(X) = 2H_1^{(1)}(X)\,. \tag{28}$$

Thus the equalities (24)—(28) and Lemma 6 imply (22).

7. **Lemma 8.** The equality

$$\tilde{I} = \int_{E_{n-1}} (mx_1 - x_n)[(y_1-x_1)^2 + \dots + (y_{n-1}-x_{n-1})^2 + (my_1-x_n)^2]^{-\frac{n}{2}} dY = \tfrac{1}{2}\theta_n$$

holds.

Proof. We apply the transformation

$$y_j - x_j = (mx_1 - x_n) r \varphi_j(\omega) \qquad j = 1, \dots, n-1$$

Then

$$\tilde{I} = \int_H K_{n-1}(\omega)\, d\omega \int_0^\infty r^{n-2}\{r^2 + [mr\varphi_1(\omega) + 1]^2\}^{-\frac{n}{2}} dr\,,$$

where $\varphi_1(\omega) = \prod_{i=1}^{n-1} \cos\omega_i$.

We shall first compute the integral

$$\tilde{I}_1(\omega) = \int_0^\infty r^{n-2}[r^2 + (mr\varphi_1 + 1)^2]^{-\frac{n}{2}} dr\,.$$

Applying the substitution $r = \dfrac{1}{t}$ and $t + m\varphi_1 = z$ we obtain

$$\tilde{I}_1(\omega) = \int_{m\varphi_1}^\infty (z^2+1)^{-\frac{n}{2}} dz$$

so, by (16)

$$(29)\quad \begin{cases} \widetilde{I}_1(\omega) = -\Big\{\frac{1}{n-2} m\varphi_1(1+m^2\varphi_1^2)^{-\frac{n}{2}+1} + \frac{n-3}{n-2}\cdot\frac{1}{n-4} m\varphi_1(1+m^2\varphi_1^2)^{-\frac{n}{2}+2} + \\ \qquad + \ldots + \frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\ldots\cdot\frac{2}{3}\cdot 1\left[m\varphi_1(1+m^2\varphi_1^2)^{-\frac{1}{2}} - 1\right]\Big\} \quad \text{for odd } n \\ \widetilde{I}_1(\omega) = -\Big\{\frac{1}{n-2} m\varphi_1(1+m^2\varphi_1^2)^{-\frac{n}{2}+1} + \frac{n-3}{n-2}\cdot\frac{1}{n-4} m\varphi_1(1+m^2\varphi_1^2)^{-\frac{n}{2}+2} + \\ \qquad + \ldots + \frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdot\ldots\cdot\frac{3}{4}\cdot\frac{1}{2}\left[\operatorname{arctg}(m\varphi_1) - \frac{\pi}{2}\right]\Big\} \quad \text{for even } n\,. \end{cases}$$

We have for every positive integer p

$$\int_0^{2\pi} m\varphi_1(1+m^2\varphi_1^2)^{-\frac{p}{2}} d\omega_1 = 0$$

so

$$\int_H K_{n-1}(\omega)\, m\varphi_1(1+m^2\varphi_1^2)^{-\frac{p}{2}} d\omega = 0\,.$$

We shall now compute the integral

$$K = \int_H K_{n-1}(\omega) \operatorname{arctg}(m\varphi_1)\, d\omega\,.$$

Let $c = m\cos\omega_1\cos\omega_3 \ldots \cos\omega_{n-2}$. We have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\omega_2 \operatorname{arctg}(c\cos\omega_2)\, d\omega_2 = \sin\omega_2 \operatorname{arctg}(c\cos\omega_2)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + c\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2\omega_2}{1+c^2\cos^2\omega_2} d\omega_2\,,$$

so

$$K = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\omega_2 \ldots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\omega_{n-2} \int_0^{2\pi} K_{n-1}(\omega)\, m \sin^2\omega_2 \cos\omega_3 \ldots \cos\omega_{n-2}\, \Gamma(\omega)\, d\omega_1 = 0\,,$$

where $\Gamma(\omega) = \dfrac{\cos\omega_1}{1+c^2\cos^2\omega_2}$.