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On the Existence of Maximal Solutions of Some Functional Equations

The present paper deals with certain applications of some properties of the increasing function defined and ranged in the same partially ordered space in order to consider the existence of the maximal solutions of some functional equations.

1. Let $(U, \leqslant)$ be an ordered space, such that for each non-empty subset U' of U, there exists $\sup U'$ in U. Let Z be a non-empty set and let W be a non-empty family of subsets of Z, such that $\emptyset \notin W$ and if $Z_1, Z_2 \in W$ then $Z_1 \cap Z_2 \in W$. In the set M of all mappings from Z into U we consider a relation $\leqslant_W^*$ defined as follows: if $u_1, u_2 \in M$ then $u_1 \leqslant_W^* u_2$ if and only if there exists $Z' \in W$ such that $u_1(z) \leqslant u_2(z)$ for $z \in Z'$. We shall still write $\leqslant^*$ instead of $\leqslant_W^*$ for convenience, but this relation depends obviously on the family W. We associate with the relation $\leqslant^*$ an equivalence relation $=^*$, that is if $u_1, u_2 \in M$ then $u_1 =^* u_2$ if and only if $u_1 \leqslant^* u_2$ and $u_2 \leqslant^* u_1$. Let us denote by M^* the factor set $M/=^*$; that is, the set of all equivalence classes of u belonging to M; in other words: $u^* \in M^*$ if and only if by the definition $u^* = [u^*]$ with some $u \in M$. In M^* we consider an ordering relation $\preccurlyeq$ defined as follows: if $u^*, v^* \in M^*$, then $u^* \preccurlyeq v^*$ if and only if $u \leqslant^* v$, for some $u, v \in M$ and $u \in u^*$; $v \in v^*$. It is obvious that it is really an ordering relation, that is

1° $u^* \preccurlyeq u^*$ for each $u^* \in M^*$,
2° if $u^* \preccurlyeq v^*$ and $v^* \preccurlyeq u^*$, then $u^* = v^*$,
3° if $u^* \preccurlyeq v^*$ and $v^* \preccurlyeq w^*$, then $u^* \preccurlyeq w^*$.

Let P be a certain subclass of M and P^* be the factor class $P/=^*$ (of course $P^* \subset M^*$).

Assume that:

(i) For each subset C of P the mapping

$$\hat{u}_C\colon\ Z \ni z \mapsto \sup\{u(z)\colon\ u \in C\}$$

is an element of P.

If

$$F : Z \times P \ni (z, u) \mapsto F(z, u) \in U$$

is a given mapping, then for each $u \in P$ we associate with F a mapping $F\langle u\rangle$ defined by the formula:

$$F\langle u\rangle : Z \ni z \mapsto F\langle u\rangle(z) \stackrel{\mathrm{df}}{=} F(z, u) \in U$$

and for each $z \in Z$ a mapping $F[z]$ defined by the formula:

$$F[z] : P \ni u \mapsto F[z](u) \stackrel{\mathrm{df}}{=} F(z, u) \in U .$$

Assume that:
(ii) For every $z \in Z$ the mapping $F[z]$ is "increasing" in the sense that $u_1 \leqslant^* u_2 \Rightarrow$ $\Rightarrow F[z](u_1) \leqslant F[z](u_2)$
and
(iii) for each $u \in P$ the mapping $F\langle u\rangle$ belongs to P.
Further we put:

$$\widetilde{F} : P \ni u \mapsto F\langle u\rangle \in P$$

and

$$\widetilde{F}^* : P^* \ni u^* \mapsto F\langle u\rangle^* \in P^* .$$

2. Theorem 1. Under the above assumptions (and with the above notations) from section 1 if a set $Q^* \subset P^*$ defined by the following formula

$$Q^*\{u^* \in P^* : u^* \preccurlyeq \widetilde{F}^*(u^*)\}$$

is not empty then the equation $u^* = \widetilde{F}^*(u^*)$ has in P^* a maximal solution.

Proof: The mapping $\widetilde{F}^*$ is increasing with respect to the ordering relation $\preccurlyeq$ in P^*, since the assumption (ii) implies that for each $z \in Z$ and $u_1, u_2 \in P$ we have
$F\langle u_1\rangle(z) = F[z](u_1) \leqslant F[z](u_2) = F\langle u_2\rangle(z)$ if $u_1 \leqslant^* u_2$; hence if $u_1^*, u_2^* \in P^*$ and $u_1^* \preccurlyeq u_2^*$ then $F\langle u_1\rangle^* \preccurlyeq F\langle u_2\rangle^*$. From the assumption (i) it follows that there exists in P^* a $\sup Q^*$ defined by the formula

$$\sup Q^* = [\hat{u}_Q]^* \text{ (= the equivalence class of } \hat{u}_Q) .$$

In order to finish the proof the following theorem of A. Pelczar [1] may be applied:

Theorem 2. Let P be a non-empty partially ordered set, let v be an increasing map of P into P, further let the subset Q of the set P defined by the formula

$$Q = \{z \in P : z \leqslant v(z)\}$$

be non-empty and let there exists in P $\sup Q$ (where by $\sup Q$ we shortly denote the least upper bound of Q in P).

Then $\hat{z} = \sup Q$ is the maximal solution of the equation $z = v(z)$ in P, i.e. such a solution that for each solution $z \in P$ of the equation $z = v(z)$ we have $z \leqslant \hat{z}$.

3. We shall give some application of theorem 1. Let S be the family of the real functions $u = u(t, x)$ defined for $0 \leqslant t \leqslant t_0$ and almost every x in (a, b), which have the following properties:

1° $u(t, x)$ is Lebesgue integrable with respect to x,

2° $u(t, x)$ is absolutely continuous in t for almost every x,

3° $[\int_a^b u(t, y)dy]^{-1}$ is integrable with respect to t in every compact interval contained in $0 \leqslant t < t_0$.

Let us also assume:

Assumption H_1. A function $v = v(x)$ is non-negative Lebesgue integrable in (a, b) and $\int_a^b v(x)dx > 0$,

Assumption H_2. We assume that a function $\lambda = \lambda(t, x)$ is defined and non-negative almost everywhere in $\{(t, x): 0 < t, a < x < b\}$. In every finite rectangle $0 < t < t_1$, $a < x < b$ is bounded and Lebesgue integrable as the function of two variables.

Under these assumptions J. Szarski [2] proved the equivalence of the following two problems:

Problem P1. To find in S a solution of the integro-differential equation

$$[\int_a^b u(t, y)dy]u_t'(t, x) = -\lambda(t, x)u(t, x)$$

satisfying the initial condition $u(0, x) = v(x)$ for almost every x.

Problem P2. To find in S a solution of the equation

$$u(t, x) = v(x)\exp\left\{-\int_0^t \left[\lambda(s, x)/\int_a^b u(s, y)dy\right]ds\right\},$$

where λ and v are as in the assumptions H_1 and H_2.

J. Szarski also proved that for a sufficiently small t_0 there exists a solution of problem P2 in $0 \leqslant t < t_0$, $a < x < b$ and, moreover, every two solutions are equal for almost every x. The method used in [2] was that of successive approximations. Under all the above assumptions with a sufficiently small t_0, we can apply Theorem 1 in order to show the existence of a solution of a certain generalization of problem P2.

Choose $t_0 > 0$ so small that $\exp(-2t_0\Lambda_0/A_0) > 1/2$, $t_0 < 1$, where $A = \int_a^b v(y)dy$ and $\lambda(t, x) < \Lambda_0$ for $0 \leqslant t \leqslant t_0$, $a < x < b$. Then the assumptions of Theorem 1 are fulfilled for:

$Z = \{z = (t, x): 0 \leqslant t < t_0, a < x < b\}$,

$U = R^1$,

$W =$ the family of the subsets Z' of Z such that $Z' \in W \Leftrightarrow z = (t, x) \in Z'$ for $t \in [0, t_0)$ and almost every $x \in (a, b)$,

$P =$ the family of the functions $u \in S$ such that if $u \in P$ then $v(x)/2 \leqslant u(t, x) \leqslant$ $\leqslant v(x)$ for $0 \leqslant t < t_0$ and almost every $x \in (a, b), F : Z \times P \ni (t, x, u) \mapsto F(t, x, u)$ $= v(x)\exp\{-\int\limits_0^t [\lambda(s, x)/\int\limits_a^b K(s, y, u(s, y))dy]ds\}$ with the function $K : Z \times$ $\times P \ni (t, x, u) \mapsto K(t, x, u(t, x)) \in U$ such that
1° $K(\cdot, \cdot, u) \in P$ for each fixed $u \in P$,
2° if $u_1(t, x) \leqslant u_2(t, x)$ for almost every $(t, x) \in Z$ then the inequality $K(t, x, u_1(t, x)) \leqslant K(t, x, u_2(t, x))$ is fulfilled for $u_1, u_2 \in P$ and almost every $(t, x) \in Z$.

We have $v/2 \in Q$ since $v(x)\exp\{-\int\limits_0^t [\lambda(s, x)/\int\limits_a^b K(s, y, v(y)/2)dy]ds\} \geqslant$ $\geqslant v(x)\exp\{-\int\limits_0^t [\lambda(s, x)/\int\limits_a^b (v(y)/2)dy]ds\} \geqslant v(x)/2$ for $0 \leqslant t \leqslant t_0$, $a < x < b$. Hence Q is not empty. Then Q^* is not empty and there exists a maximal solution of the equation $u^* = \tilde{F}^*(u^*)$ in P^*.

In consequence we have the following:

Theorem 3. There exists a solution $u \in S$ of the equation

$$\text{(j)} \quad u(t, x) = v(x)\exp\{-\int\limits_0^t [\lambda(s, x)/\int\limits_a^b K(s, y, u(s, y))dy]ds\}$$

with the given function v, λ, K as above.

We shall finish by some remarks on the uniqueness of the solution in S of the equation

$$\text{(jj)} \quad u(t, x) = v(x)\exp\{-\int\limits_0^t [\lambda(s, x)/\int\limits_a^b H(s, y, u(s, y))dy]ds\}$$

with the functions v, λ as in the equation (j) and a function $H : Z \times S \ni (t, x, u) \mapsto$ $\mapsto H(t, x, u(t, x)) \in U$ such that $H(\cdot, \cdot, u) \in S$ for each fixed $u \in S$.

Let a function $L : Z \times \hat{S} \ni (t, x, w) \mapsto L(t, x, w(t, x)) \in U$ be such that $L(\cdot, \cdot, w) \in \hat{S}$ for each fixed $w \in \hat{S}$, where by $\hat{S}$ we denote the family of the functions from Z to U, non-negative and Lebesgue integrable with respect to x. Suppose moreover that the following inequality $|H(t, x, u_1(t, x)) - H(t, x, u_2)|$ $\leqslant L(t, x, |u_1(t, x) - u_2(t, x)|)$ is fulfilled for almost every (t, x) such that $a < x < b$, $0 \leqslant t \leqslant t_1 < t_0$ where t_0 is sufficiently small and for any two solutions u_1, u_2 of the equation (jj).

Remark 1. The above assumptions are fulfilled in particular if $H : Z \times$ $\times R \ni (t, x, u) \mapsto H(t, x, u) \in R$ and $L : Z \times R \ni (t, x, u) \mapsto L(t, x, u) \in R$ are continuous with respect to the third variable u, measurable with respect to (t, x) and such that there are summable functions M_H and M_L from Z into R, such that $|H(t, x, u)| \leqslant M_H(t, x)$ and $|L(t, x, u)| \leqslant M_L(t, x)$ for almost every $(t, x) \in Z$.

Remark 2. Under the above assumptions on the functions H and L, if the inequality (1) $\int\limits_a^b L(t, x, w(t, x))dx \leqslant \int\limits_a^b w(t, x)dx$ or (2) $L(t, x, w(t, x)) \leqslant aw(t, x)$ with $a \in R$, $a > 1$ is fulfilled for almost every $(t, x) \in Z$, then for $0 \leqslant t \leqslant t_1$ we have the uniqueness of the solution in S of the equation (jj). This can easily be proved by the method used in [2].

Remark 3. If the function L does not fulfil the conditions (1) and (2), but is increasing with respect to w, then we can apply Theorem 1 in order to show the existence of a maximal solution in S of the following equation:

$$\text{(jjj)} \quad w(t,x) = f(x)\int_0^t \Big[\int_a^b L\big(s, y, w(s,y)\big)dy\Big]ds,$$

where a function $f:(a,b) \ni x \mapsto f(x) \in U$ is non-negative and Lebesgue integrable in (a,b).

For every two u_1, u_2 the solutions in S of the equation (jj) and for $f = v\Lambda_1/K^2$ where v is as in (jj), $\Lambda_1 \geqslant \lambda(t,x)$, $\int_a^b H\big(t,y,u_1(t,y)\big)dy \geqslant K > 0$ and $\int_a^b H\big(t,y,u_2(t,y)\big)dy \geqslant K > 0$ in $0 \leqslant t \leqslant t_1$ $a < x < b$, we have the evident inequality $|u_1(t,x) - u_2(t,x)| \leqslant \hat{w}(t,x)$ for almost every (t,x), where $\hat{w}$ is the maximal solution of the equation (jjj). Hence we have the following:

Theorem 4. If the equation (jjj) has in the class S the unique solution $\hat{w} \equiv 0$, then the equation (jj) has at most one solution (in the same class S).

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REFERENCES

[1] A. Pelczar, *On invariant points of monotone transformations in partially ordered spaces.* Ann. Polon. Math. 17 (1965) 49—53.

[2] J. Szarski, *On an integro-differential equation.* Ann. Polon. Math. 14 (1964) 321—333.