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Classification of Continuous Solutions of a Functional Inequality

1. Continuous solutions $\psi\colon \langle 0, a) \to \boldsymbol{R}^+$ of the linear homogeneous inequality

$$\psi[f(x)] \leqslant g(x)\psi(x) \tag{1}$$

have been dealt with by the first author of the present paper in [1] under the following general hypotheses

(*H*) *The function* $f\colon \langle 0, a) \to \langle 0, a)$ *is continuous and strictly increasing in* $\langle 0, a)$, *and* $0 < f(x) < x$ *in* $(0, a)$. *The function* $g\colon \langle 0, a) \to (0, +\infty)$ *is continuous in* $\langle 0, a)$.

Given a solution $\psi \in C\langle 0, a)$ of inequality (1) we may be interested in finding a function $\varphi \in C\langle 0, a)$ fulfilling the functional equation

$$\varphi[f(x)] = g(x)\varphi(x) \tag{2}$$

for $x \in \langle 0, a)$ and such that

$$\psi(x) \geqslant \varphi(x) \quad \text{for } x \in \langle 0, a)\,. \tag{3}$$

This problem has also been considered in [1], but — in the case where equation (2) has a *C*-solution depending on an arbitrary function — the results are not very effective.

In the present paper we shall study inequality (1) in a class of functions that behave near the origin like a power function with an exponent $p > 0$, viz. in the class U^p of functions that has been introduced by M. Kuczma [5].

It turns out that, with the aid of some results due to the second author of the present paper [2], we can classify continuous solutions ψ of inequality (1) and find some formulae for the corresponding U^p-solutions φ of equation (2) that fulfils (3). Some related questions will be also considered.

2. We first complete some results on solutions of inequality (1) that have been proved in [1]. In this section we assume hypotheses (*H*) to hold.

We put

$$I = \langle 0, a), a > 0; \tag{4}$$

$$f^0(x) = x, f^{n+1}(x) = f(f^n(x)), \quad n = 0, 1, \ldots, x \in I, \tag{5}$$

$$G_n(x) = \prod_{i=0}^{n-1} g[f^i(x)], \quad n = 1, 2, \ldots, x \in I \tag{6}$$

and denote

$$\Phi = \{\varphi \in C[I] : \varphi(x) > 0 \quad \text{for } x \in (0, a)\}.$$

Lemma 1. If ψ is a Φ-solution of inequality (1) in I then there exists the limit

$$\varphi_0(x) = \lim_{n\to\infty} \psi[f^n(x)]/G_n(x) \tag{7}$$

for $x \in I$, and the function φ_0 is a solution of (2), upper semicontinuous in I and continuous at zero. Moreover, if $\varphi_0 \in \Phi$, then it is the maximal solution of (2) in Φ satisfying (3) and the function ψ can be written in the form

$$\psi(x) = \eta(x)\varphi_0(x) \quad \text{for } x \in I,$$

where $\eta \in \Phi$ is an $\{f\}$-decreasing function fulfilling further the condition $\eta(0) = 1$.

Remarks. The phrase "φ_0 is the maximal solution" has the following meaning: if, for a solution $\varphi \in \Phi$ of equation (2) in I, there exists an $x_0 \in (0, a)$ such that $\varphi(x_0) > \varphi_0(x_0)$, then there exists a positive integer k such that $\varphi[f^k(x_0)] > \psi[f^k(x_0)]$.

2. A function η: $I \to \mathbf{R}$ is said to be $\{f\}$-decreasing iff

$$\eta[f(x)] \leqslant \eta(x) \quad \text{for } x \in I.$$

A necessary and sufficient condition for the φ_0 given by (7) to belong to the class Φ is given in the next lemma.

Lemma 2. Let ψ be a Φ-solution of inequality (1) in I. Then $\varphi_0 \in \Phi$ iff there exists the limit $\lim\limits_{x\to 0+} \psi(x)/\varphi_0(x)$ (then necessarily equal to 1). Moreover, for $\varphi \in \Phi$ the limit $\lim\limits_{x\to 0+} \psi(x)/\varphi(x)$ exists and it is finite and different from zero iff $\varphi = c\varphi_0$, $c > 0$.

The last lemma in this section contains some relations between the existence of the limit

$$b = \lim_{x\to 0+} \psi[f(x)]/\psi(x) \tag{8}$$

and the property $\varphi_0 \in \Phi$. Note that if ψ fulfils (1) then we always have $b \leqslant g(0)$ if b exists.

Lemma 3. If φ_0 assigned to a Φ-solution ψ of (1) by (7) belongs to Φ, then the limit (8) exists, and

$$b = g(0). \tag{9}$$

On the other hand, if b exists and

$$b < g(0), \tag{10}$$

then $\varphi_0(x) = 0$ for $x \in I$.

Proofs of these lemmas are to be found in [1].

3. In the present paper we shall be especially interested in finding for a given ψ that satisfies (1) such a solution $\varphi \in \Phi$ of (2) which is the maximal solution fulfilling (3). The formula (7) given in lemma 1 produces such a solution in the case where equation (2) has the only one-parameter family of C-solutions in Φ. However, this is the case iff

(A) *The sequence* $G_n(x)$ *defined by* (6) *with* (5) *approaches a non-zero limit in* I *and the limit function is continuous in* I.

As has been proved in [3] (cf. also [6], Ch. 2) all the C-solutions of (2) in I are then given by the formula

$$\varphi(x) = c/\lim_{n\to\infty} G_n(x), \ c \in \boldsymbol{R}, \tag{11}$$

and the φ_0 defined by (7) we get for $c = 1$.

However in the case where

(B) *There is an interval* $J \subset I$ *such that* $\lim\limits_{n\to\infty} G_n(x) = 0$, *uniformly in* J

equation (2) possesses in I a C-solution depending on an arbitrary function (cf. [3]) and without further assumptions we cannot know whether the φ_0 given by (7) belongs to Φ or not.

Since in case (B) any solution $\varphi \in \Phi$ of equation (2) admits the value zero at $x = 0$ thus in the sequel we shall deal with solutions (both of inequality (1) and equation (2)) in a class U^p of functions, whose definition follows.

Definition (cf. [5]). A function $a \in \Phi$ is said to belong to the class U^p of functions, where $p > 0$, iff there exists the limit $\lim\limits_{x\to 0+} x^{-p}a(x)$ and it is a positive number.

In this section we present some results from the theory of asymptotic behaviour of C-solutions of equation (2), given in [2]. They will contain some conditions for the existence of the only one-parameter family of solutions of (2) in the case (B). Beside hypotheses (H) we assume here that either

(i) *There exist numbers*: $s \in (0,1)$, $\alpha > 0$, $\beta > 0$ *such that*

$$f(x) = sx + 0(x^{1+\alpha}), \quad x \to 0+, \tag{12}$$

$$g(x) = g(0) + 0(x^{\beta}), \quad x \to 0+, \tag{13}$$

or

(ii) *There exist positive numbers* m, μ, ν, q *and* r *such that*

$$f(x) = 1 - rx^{m+1} + 0(x^{m+1+\mu}), \quad x \to 0+ ,$$

$$g(x) = 1 - qx^{m} + 0(x^{m+\nu}), \quad x \to 0+ .$$

In the sequel the number $p > 0$ will be regarded as fixed.

Lemma 4 (cf. [2], Ch. 3 and 5). If hypotheses (H) and (i) with $g(0) < 1$ or (ii) are fulfilled then equation (2) has in I a C-solution depending on an arbitrary function.

If, moreover, $g(0) = s^p$ in (i), resp. $q = pr$ in (ii), then (and only then) there is the only one-parameter family $\{\varphi_t\}$, $t > 0$, of solution of (2) in the class U^p, and they are given by the formula (cf. (5) and (6))

$$\varphi_t(x) = t \cdot \lim_{n\to\infty} \{[f^n(x)]^p / G_n(x)\} , \quad x \in I . \tag{14}$$

Proof. For the case (ii) the lemma follows from theorems 5.1, 5.4, and 5.6 proved in [2]. The case (i), in fact, has also been dealt with in [2] (th. 3.3, cf. also [3]) but we assumed there that the functions f and g have asymptotic power series expansions at the origin, so that for this case there should here be given an independent proof.

Assume (i) to hold, and denote

$$\Phi_0 = \{\varphi \in C\langle 0, a) : \varphi(x) > 0 \quad \text{for } x \in \langle 0, a)\} .$$

There is one-to-one correspondence among solutions $\varphi \in U^p$ of (2) and solutions $\bar{\varphi} \in \Phi_0$ of the equation

$$\bar{\varphi}[f(x)] = \bar{g}(x)\bar{\varphi}(x) \tag{15}$$

where

$$\bar{g}(x) = \begin{cases} g(x)[f(x)/x]^{-p} & \text{for } x \in I\setminus\{0\} \\ g(0)s^{-p} & \text{for } x = 0 \end{cases} \tag{16}$$

For, given a solution $\bar{\varphi} \in \Phi_0$ of (15), put

$$\varphi(x) = x^p \bar{\varphi}(x) , \quad x \in I$$

to obtain the solution of (2) belonging to U^p, and conversely. This results directly from (2) and (12)—(16).

Thus following the preliminary remarks in this section it is enough to prove that for the sequence

$$\bar{G}_n(x) = \prod_{i=0}^{n-1} \bar{g}[f^i(x)] , \quad n = 1, 2, \ldots, x \in I \tag{17}$$

case (A) occurs. We shall prove that $\{\bar{G}_n(x)\}$ uniformly converges in every interval $\langle 0, d\rangle \subset I$.

Conditions (12) and (13) with $g(0) = s^p$ yield by (16):

$$\bar{g}(x) = [s^p + 0(x^\alpha)][s + 0(x^\beta)]^{-p} = 1 + \gamma(x)\,, \tag{18}$$

where

$$\gamma(x) = 0(x^\varrho), \quad \varrho = \min(\alpha, \beta) > 0\,.$$

Consequently, there is an $M > 0$ and a $\delta > 0$ such that

$$|\gamma(x)| < Mx^\varrho \quad \text{for } x \in \langle 0, \delta\rangle \subset I\,.$$

But from (12) it follows (cf. e.g. [7]) that there are: a $K > 0$ and a positive integer n_0 to fulfil

$$f^n(d) \leqslant Ks^n \quad \text{for } n \geqslant n_0\,.$$

This implies by hypotheses (H)

$$|\gamma[f^n(x)]| < M[f^n(d)]^\varrho < KM^\varrho s^{n\varrho}$$

for n sufficiently large and every $x \in \langle 0, d\rangle$. From (18) we see that the infinite product $\prod\limits_{n=0}^{\infty} \bar{g}[f^n(x)]$ uniformly converges in $\langle 0, d\rangle$, so that its limit is a function belonging to Φ_0.

All solutions $\bar{\varphi} \in \Phi_0$ of equation (15) are then given by the formula (cf. (11))

$$\bar{\varphi}(x) = t/\lim_{n\to\infty} \bar{G}_n(x)\,,$$

which through (15) and (17) takes the form (14).

If in (i) we have $1 > g(0) \neq s^p$ then by (16) there is $\bar{g}(0) \neq 1$ which implies (cf. [6], th. 2.2.—2.4, p. 48 and 51) that equation (15) has either only a trivial C-solution in I or that the solution depends on an arbitrary function. In any case the C-solution of (15) in I admits the value zero at the origin, so that the equation does not have solutions in the class Φ_0. Thus (2) has no solution in the class U^p either. This ends the proof of the lemma.

4. Now we shall obtain more information on the solutions $\psi \in U^p$ of inequality (1) with the aid of the results presented in sections 2 and 3.

Theorem 1. Assume (H) and (i) or (ii) to hold. If ψ is a solution of inequality (1) belonging to the class U^p, then the φ_0 defined by (7) belongs to U^p also, and it is the maximal solution of equation (2) which fulfils (3).

Proof. If $\psi \in U^p$ then there exists a function, λ say, $\lambda \in \Phi_0$ such that

$$\psi(x) = x^p \lambda(x),\ x \in I\,. \tag{19}$$

Thus we get for (7):

$$\psi[f^n(x)]/G_n(x) = \{[f^n(x)]^p/G_n(x)\}\,\lambda[f^n(x)]\,.$$

From lemma 4 it follows that the sequence in brackets approaches a limit $\varphi_1 \in U^p$ (cf. (14)). Since λ is continuous at zero and (H) implies $\lim\limits_{n\to\infty} f^n(x) = 0$ for any $x \in I$ then the limit (7) takes the form

$$\varphi_0(x) = \lambda(0)\varphi_1(x),\ x \in I\ , \tag{20}$$

and $\varphi_0 \in U^p \subset \Phi$. The inequality (3) with φ_0 as the maximal follows from lemma 1, and the theorem is proved.

We may also complete the result quoted as lemma 3.

Theorem 2. Assume (H) to hold, let $\psi \in U^p$ be a solution of (1) and let φ_0 be defined by (7) for the ψ. Then

1° if $f \in U^1$, then the limit (8) exists, and

$$b = [f'(0)]^p\ , \tag{21}$$

2° if $\varphi_0 \in U^p$, then $b = g(0)$.

3° if $b = g(0)$ and (i) is satisfied, then $\varphi_0 \in U^p$.

4° if (i) holds and

$$\varphi_0(x) = 0 \quad \text{for } x \in I\ , \tag{22}$$

then $b < g(0)$.

Proof. 1°. As $\psi \in U^p$ we can use (19) that yields for (8):

$$b = \lim_{x\to 0+} \frac{\psi[f(x)]}{\psi(x)} = \lim_{x\to 0+} \left[\frac{f(x)}{x}\right]^p \frac{\lambda[f(x)}{\lambda(x)} = [f'(0)]^p\ .$$

2°. Since $U^p \subset \Phi$, then for $\varphi_0 \in U^p$ equality $b = g(0)$ follows from lemma 3.

3°. It follows from (12) of (i) that $f \in U^1$ and $f'(0) = s$, i.e. by (21) we have $b = g(0) = s^p$. Lemma 4 thus yields the existence of the limit $\varphi_1 \in U^p$ (cf. (14)), so that φ_0 can be written in form (20), giving $\varphi_0 \in U^p$.

4°. Now we also have $f'(0) = s$ and (cf. (7) and (19))

$$0 = \varphi_0(x) = \lim_{n\to\infty} \frac{\psi[f^n(x)]}{G_n(x)} = \lim_{n\to\infty} \frac{[f^n(x)]^p}{G_n(x)} \lambda[f^n(x)]$$

for $x \in I$. Since the sequence (17) can be written in the form

$$\bar{G}_n(x) = x^p G_n(x) [f^n(x)]^{-p} \quad \text{for } x \in I\setminus\{0\}\ , \tag{23}$$

as follows from (16), then we see that $\bar{G}_n(x) \to +\infty$ for $x \in (0, a)$. But this implies that for this sequence neither (A) nor (B) occurs. By (i) and lemma 4 it is seen that necessarily $g(0) \neq s^p$. On the other hand we always have $b \leqslant g(0)$, thus (21) yields $b < g(0)$. The proof is completed.

Theorem 2 claims that if (H) and (i) are assumed then (9) holds iff $\varphi_0 \in U^p$ as well as (10) holds iff (22) is valid. This is no longer true if we assume (ii) instead of (i), i.e. if we deal with the indeterminate case $g(0) = f'(0) = 1$.

Theorem 3. Let $\psi \in U^p$ be a solution of (1) and let φ_0 be given by (7). If (H) and (ii) are fulfilled (the last with a q, $0 < q < pr$) then relation (22) holds true.

Proof. First note that by (ii) we have $f \in U^1$ and $f'(0) = 1$, so that by (21):

$$b = 1 = g(0)\,. \tag{24}$$

Formulae (7), (19) and (23) yield for $x \in (0, a)$

$$\varphi_0(x) = \lambda(0) \lim_{n\to\infty} [f^n(x)]^p / G_n(x) = \lambda(0) \lim_{n\to\infty} [\bar{G}_n(x)]^{-1} x^p\,. \tag{25}$$

Thus we are led to examine sequence (17), i.e. the function (16). Conditions (ii) being assumed we obtain the relation

$$\bar{g}(x) = \big(1 - qx^m + 0(x^\nu)\big)\big(1 - rx^m + 0(x^\mu)\big)^{-p} = 1 + (pr - q)x^m + 0(x^{m+\varkappa})$$

as $x \to 0+$, where $\varkappa = \min(m, \mu, \nu)$. Since (cf. [7]) the sequence $\{f^n(x)\}$ now behaves at infinity like $k(x) n^{-1/m}$, $k(x) > 0$, then for any $x \in (0, a)$ there is a $K(x) > 0$ such that $K(x) n^{-1} < \bar{g}[f^n(x)] - 1$ at least for n sufficiently large. Consequently

$$\lim_{n\to\infty} \bar{G}_n(x) = +\infty \text{ in } (0, a)\,. \tag{26}$$

Moreover, we have $\bar{G}_n(0) = 1$, so that (26) and (25) imply (22), and the proof is completed.

If the hypotheses of theorem 3 are satisfied then from lemma 4 it follows that equation (2) does not possess any solution in the class U^p. In the example that concludes the paper we show that it is not so for solutions of inequality (1).

Example. Consider the inequality

$$\psi\left(\frac{x}{1+x}\right) \leqslant (1-x)\psi(x)\,, \qquad x \in I := \langle 0, \tfrac{1}{2})\,. \tag{27}$$

Here the given functions f and g fulfil hypotheses (H) and (ii) with $m = r = q = 1$, $a = 1$ and an arbitrary $\beta > 0$. Taking e.g. $p = 2$, we have $q < pr$, and inequality (27) has in the class U^2 the solutions $\psi(x) = cx^2$, $x \in I$, $c > 0$; since $(1+x)^{-2} \leqslant 1 - x$ for $x \in I$.

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