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On the Rational Choice of Coordinate Functions in the Least-Square Method

Introduction. The purpose of this note is to give a mode of choosing coordinate functions in the least-square method. In any case we may receive a result similar to that in paper [1], where a choice of coordinate functions is carried out by the Bubnov-Galerkin method. As in paper [2] we may also estimate the rate of convergence of the approximation sequence of this method. Under any assumptions we shall prove that the approximation sequence of the least-square method converges more quickly than the approximation sequence of the Bubnov-Galerkin method. The results of this note will be illustrated by one example.

§ 1. CONVERGENCE OF THE LEAST-SQUARE METHOD

Let H be a Hilbert separable space and let A and B be the linear self-adjoint positive definite operators in H, such that $D(A) = D(B)$.

Lemma 1. If the operators A and B satisfy the above conditions then $A^{-1}B^2A^{-1}$ and $B^{-1}A^2B^{-1}$ are bounded operators.

Proof. It follows from the assumptions of Lemma 1 that AB^{-1}, BA^{-1}, $A^{-1}B$ and $B^{-1}A$ are bounded operators in H (cf. [4] th. 3.1). Since $A^{-1}B^2A^{-1} = (A^{-1}B)(BA^{-1})$ and $B^{-1}A^2B^{-1} = (B^{-1}A)(AB^{-1})$, therefore the thesis of Lemma 1 follows from this.

Lemma 2. If G is a self-adjoint positive definite operator and C is a bounded linear operator in H such that $C(D(G)) \subset D(G)$, then GCG^{-1} is the bounded operator in H.

The proof of this Lemma is shown in paper [3].

Lemma 3. Under the assumptions of Lemma 1 $A^2B^{-2}, B^2A^{-2}, A^{-2}B^2$ and $B^{-2}A^2$ are bounded operators in H.

Proof. Let us observe that $A^2B^{-2} = B(B^{-1}A^2B^{-1})B^{-1}$. If we denote $C = B^{-1}A^2B^{-1}$ and $G = B$ we see that the operators C and G satisfy all the assumptions of Lemma 2. From this by Lemma 2 it follows that A^2B^{-2} is bounded operator in H. The proof that $B^{-2}A^2$ is a bounded operator is analogous. The operators $A^{-2}B^2$ and $B^{-2}A^2$ are bounded as conjugal operators to the operators B^2A^{-2} and A^2B^{-2}, respectively.

Lemma 4. Under the assumptions of Lemma 1 there exist constants c_1 and c_2 such that

$$c_1^2(B^2u, u) \leqslant (A^2u, u) \leqslant c_2^2(B^2u, u) \qquad u \in D(A^2) \cap D(B^2)\,. \tag{1}$$

Proof. It follows from the assumption that there exist constants c_1 and c_2 such that

$$c_1\|Bu\| \leqslant \|Au\| \leqslant c_2\|Bu\| \qquad u \in D(A)\,. \tag{2}$$

Since $D(A^2) \subset D(A)$ and $D(B^2) \subset D(B) = D(A)$, therefore the inequality is satisfied for arbitrary $u \in D(A^2) \cap D(B^2)$. Raising to the square the inequality (2), and observing that $\|Au\|^2 = (Au, Au) = (A^2u, u)$ and $\|Bu\|^2 = (Bu, Bu) = (B^2u, u)$, we get the inequality (1).

We still assume that there exist a denumerable sequence of eigenvalues $\{\lambda_n\}$ of operator B and a corresponding sequence of eigenvectors $\{\varphi_n\}$ which form a complete system in H. We also assume that $\{\varphi_n\}$ is an orthonormal system in H and that $\lim \lambda_n = +\infty$.

Let us consider the operational equation

$$Au = f \tag{3}$$

where A is an operator satisfying the above conditions and $f \in D(A)$. Let us denote

$$u_n = \sum_{k=1}^{n} a_k\varphi_k \tag{4}$$

the nth approximation of the solution of equation (3) in the sense of the least-square method. Let u_0 denote the solution of equation (3). We assume that u_0 exists.

We shall prove the following

Theorem 1. If $\varphi_1, \ldots, \varphi_n$ in formula (4) are the eigenvectors of the operator B, then $A^2u_n \to A^2u_0$ by norm of the space H.

Proof. It is known that the coefficients in formula 4 satisfy the following system of equations

$$\sum_{k=1}^{n} a_k(A\varphi_k, A\varphi_j) = (f, A\varphi_j) \qquad j = 1, \ldots, n\,. \tag{5}$$

Since A is a self-adjoint operator, so the system (5) may be written in the form

$$\sum_{k=1}^{n} a_k(A^2\varphi_k, \varphi_j) = (Af, \varphi_j) \qquad j = 1, \ldots, n\,. \tag{6}$$

System (6) is that of Bubnov-Galerkin for the equation

$$A^2u = Af\,. \tag{7}$$

By Lemmas 1, 2, 3 and 4 the operators A^2 and B^2 satisfy all the assumptions of Theorem 1 from paper [1], therefore in virtue of this theorem we obtain $A^2u_n \to A^2u_0$.

§ 2. THE RATE OF CONVERGENCE OF APPROXIMATION IN THE SENSE OF THE LEAST-SQUARE METHOD

In this section the rate of convergence of the sequence $\{u_n\}$ to u_0, under the assumptions of previous section will be estimated.

We shall prove the following

Theorem 2. If the operators A and B satisfy the previous assumptions, then

$$\|u_n - u_0\| = 0(\lambda_{n+1}^{-2})\,, \tag{8}$$

where $\{u_n\}$ and u_0 have already been and $\{\lambda_n\}$ is the sequence of eigenvectors of the operator B.

Proof. By Theorem 1 $\delta_n = A^2u_n - Af \to 0$ as $n \to \infty$. Let us observe that

$$\delta_n = \sum_{k=1}^{\infty} (\delta_n, \varphi_k)\varphi_k\,. \tag{9}$$

By orthogonalization of $\{\varphi_n\}$ and by (9) we get

$$\|\delta_n\| = \Big\{\sum_{k=1}^{\infty} |(\delta_n, \varphi_k)|^2\Big\}^{\frac{1}{2}}\,.$$

On the other hand by (6) it is obvious that

$$(\delta_n, \varphi_k) = (A^2u_n - Af, \varphi_k) = 0 \qquad \text{for} \qquad k = 1, \ldots, n\,.$$

This denotes that

$$\delta_n = \Big\{\sum_{k=n+1}^{\infty} |(\delta_n, \varphi_k)|^2\Big\}^{\frac{1}{2}}\,.$$

Now we shall estimate $\|B^{-2}\delta_n\|$. We get

$$\|B^{-2}\delta_n\|^2 = \sum_{k=1}^{\infty} |(B^{-2}\delta_n, \varphi_k)|^2 = \sum_{k=1}^{\infty} |(\delta_n, B^{-2}\varphi_k)|^2$$

$$= \sum_{k=1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k^4} = \sum_{k=n+1}^{\infty} \frac{|(\delta_n, \varphi_k)|^2}{\lambda_k^4} \leqslant \frac{\|\delta_n\|^2}{\lambda_{n+1}^4},$$

therefore

$$\|B^{-2}\delta_n\| \leqslant \frac{\|\delta_n\|}{\lambda_{n+1}^2}. \tag{10}$$

By (10) we get

$$\|u_n - u_0\| = \|A^{-2}B^2B^{-2}\delta_n\| \leqslant \|A^{-2}B^2\|\,\|B^{-2}\delta_n\| \leqslant \frac{\|A^{-2}B^2\|\,\|\delta_n\|}{\lambda_{n+1}^2}.$$

Since $A^{-2}B^2$ by Lemma 3 is a bounded operator, therefore by the last inequality we have (8). The proof is completed.

Remark 1. The estimation of rate convergence made by formula (8) generalizes a suitable estimation from paper [2]. In paper [2] using the Bubnov-Galerkin method and making an appropriate choice of the coordinate functions we received the estimation

$$\|u_n - u_0\| = 0(\lambda_n^{-1}).$$

In general, the last estimation is worse than the estimation made by formula (8).

This fact we shall illustrate by the following.

Example 1. We shall consider an ordinary differential equation of the second order

$$Au = -\frac{d}{dx}\left[p(x)\frac{du}{dx}\right] + r(x)u = f(x); \tag{11}$$

with the boundary condition

$$u(0) = u(1) = 0. \tag{12}$$

The domain $D(A)$ of the self-adjoint operator A is dense in the space $H = L_2([0,1])$. This domain $D(A)$ is a set of the functions defined in $[0,1]$ and satisfies the following conditions

1° If $u \in D(A)$ then u is the class C^2 in $[0,1]$

2° If $u \in D(A)$ then $u(0) = u(1) = 0$

3° If $u \in D(A)$ then $Au \in L_2([0,1])$.

We also assume that the functions $p(x)$, $p'(x)$ and $r(x)$ are continuous in $[0, 1]$, $p(x) \geqslant p_0 > 0$, $r(x) \geqslant -\lambda$, where λ is the smallest eigenvalue of the equation

$$-\frac{d}{dx}\left[p(x)\frac{du}{dx}\right]+\lambda u = 0$$

with boundary condition (12).

We shall prove that the operator A is positive definite in $D(A)$. Indeed

$$(Au, u) = \int_0^1 \left[-u\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+r(x)u^2\right]dx \, .$$

Integrating by-parts over the interval $[0, 1]$, and using conditions (12) we find

$$(Au, u) = \int_0^1 \left[p(x)\left(\frac{du}{dx}\right)^2+r(x)u^2\right]dx \, .$$

If $r(x) > 0$ then

$$(Au, u) \geqslant r_0 \int_0^1 u^2 dx = r_0\|u\|^2 \, ,$$

where $r_0 = \min\limits_{[0,1]} r(x)$.

If $r(x) = 0$ then

$$(Au, u) \geqslant p_0 \int_0^1 \left(\frac{du}{dx}\right)^2 dx = p_0\|u'\|^2 \, , \tag{13}$$

and is therefore sufficient to estimate $\|u'\|$.

Since $u(0) = 0$, so

$$u(x) = \int_0^x u'(t)\,dt.$$

Now, the Cauchy-Buniakovski-Schwarz inequality leads to the estimate

$$u^2(x) = \left(\int_0^x u'(t)\,dt\right)^2 \leqslant x\int_0^x [u'(t)]^2 dt \leqslant x\int_0^1]u'(t)]^2 dt = x\|u'\|^2 \, .$$

Integrating both sides over the interval $[0, 1)$ we get

$$\|u\|^2 \leqslant \tfrac{1}{2}\|u'\|^2 \, . \tag{14}$$

From (13) and (14) we obtain

$$(Au, u) \geqslant \gamma^2\|u\|^2 \text{ where } \gamma^2 = 2p_0 \, .$$

As the operator B we take $B = -\dfrac{d^2}{dx^2}$ with boundary condition (12). It is obvious that the operator B is self-adjoint and positive definite and that $D(B) = D(A)$.

The eigenfunctions of the operator B have the form

$$\varphi_n(x) = c_n \sin n\pi x\,, \quad n = 1, 2, 3, \ldots$$

where we find the coeffitients c_n from the conditions

$$\|\varphi_n\|^2 = \int_0^1 c_n^2 \sin^2 n\pi x\, dx = 1\,, \quad n = 1, 2, 3, \ldots$$

whence $c_n = \sqrt{2}$, $n = 1, 2, 3, \ldots$
Therefore we have

$$\varphi_n(x) = \sqrt{2} \sin n\pi x\,, \quad n = 1, 2, 3, \ldots$$

and the corresponding eigenvalues of the operator B are

$$\lambda_n = \pi n^2\,, \quad n = 1, 2, 3, \ldots$$

If $u_n(x) = \sum_{k=1}^{n} a_k \sin k\pi x$ is the nth approximation of the solution of equation (11) with boundary condition (12) by the least-square method and u_0 is the solution of this problem, then by Theorem 2 we get

$$\|u_n - u_0\| = 0(n^{-4})\,. \tag{15}$$

Remark 2. The estimation (15) for this problem is better than that in [4] p. 162. The estimation in [4] was obtained by the Ritz method.

REFERENCES

[1] J. Bochenek, *On the reasonable choice of the coordinate functions in the Bubnov-Galerkin method,* Ann. Soc. Math. Polon. s. I, Comm. Math. XVII 9—17 (1973).

[2] —, *Some remarks on the convergence of Ritz and Galerkin methods,* Ann. Soc. Math. Polon. s. I Comm. Math. XVII 17—28 (1973).

[3] —, *Te More general sufficient conditions for the convergence of the Bubnov-Galerkin method* (in press).

[4] С. Г. Михлин, *Численная реализациа вариационных методов,* Moskwa 1966.