ZESZYTY NAUKOWE UNIWERSYTETU JAGIELLOŃSKIEGO
CCCCIII PRACE MATEMATYCZNE, ZESZYT 17 1975.

Jan Musiałek

A Boundary Problem of Riquier Type Concerning the p-harmonic Equation in the Half-Space

Let us consider the p-harmonic equation

$$\Delta^p u(x_1, x_2, x_3) = 0 \tag{1}$$

in the half space $x_3 > 0$ where p is an integer greater than 2. We shall deal with a boundary condition of Riquier type

$$\Delta^i u(x_1, x_2, 0) = f_i(x_1, x_2) \tag{2}$$

for $i = 0, 1, \ldots, p-1$.

In the sequence $X = (x_1, x_2, x_3)$, $Y = (y_1, y_2, y_3)$ will denote two points such that $x_3 > 0, y_3 \geqslant 0$; we shall also write

$$r^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2 ,$$

$$r_1^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 + x_3)^2 .$$

Theorem 1. The Green function for (1), (2) satisfying the boundary conditions

$$\Delta^i G(X, Y)\big|_{y_3=0} = 0 , \quad i = 0, \ldots, p-1 \tag{3}$$

is of the form

$$G(X, Y) = r_1^{2p-3} - r^{2p-3} \tag{4}$$

Proof. The function defined (4) obviously satisfies equation (1). The function $G(X, Y)$ satisfies boundary conditions (3) since

$$\begin{aligned}
\Delta_Y G(X, Y)\big|_{y_3=0} &= -(2p-2)(2p-3)(r^{2p-5}-r_1^{2p-5})\big|_{y_3=0} = 0\\
&\vdots\\
\Delta_Y^{p-3} G(X, Y)\big|_{y_3=0} &= -(2p-2)(2p-3)(2p-4)\ \dots\ 7\cdot 6(r^3-r_1^3)\big|_{y_3=0} = 0\\
\Delta_Y^{p-2} G(X, Y)\big|_{y_3=0} &= -(2p-2)(2p-3)\ \dots\ 5\cdot 4(r-r_1)\big|_{y_3=0} = 0\\
\Delta_Y^{p-1} G(X, Y)\big|_{y_3=0} &= -(2p-2)(2p-3)(3p-4)\ \dots\ 2\cdot 1(r^{-1}-r_1^{-1})\big|_{y_3=0} = 0
\end{aligned}$$

and since $y_3 = 0$ implies $r = r_1$.

We shall prove that the function $u(X)$ of the form

$$u(X) = \frac{1}{4\pi}\sum_{i=0}^{p-1}\iint_{E_2} f_i(y_1, y_2) D_{y_3}\Delta_Y^{p-i-1} G(X, Y)\big|_{y_3=0}\, dy_1\, dy_2 \tag{5}$$

$G(X, Y)$ being defined by (4) is a solution of the boundary conditions (1), (2).

Let $K(X, Y) = [(y_1-x_1)^2+(y_2-x_2)^2+x_3^2]^{\frac{1}{2}}$, then

$$\begin{aligned}
D_{y_3} G(X, Y)\big|_{y_3=0} &= 2(2-3)x_3[K(X, Y)]^{2p-5},\\
D_{y_3}\Delta_Y G(X, Y)\big|_{y_3=0} &= 2(2p-2)(2p-3)(2p-5)x_3[K(X, Y)]^{2p-7}, \dots,\\
D_{y_3}\Delta_Y^{p-3} G(X, Y)\big|_{y_3=0} &= 2(2p-2)(2p-3)(2p-4)\ \dots\ 7\cdot 6x_3K(X, Y),\\
D_{y_3}\Delta_Y^{p-2} G(X, Y)\big|_{y_3=0} &= 2(2p-2)(2p-3)(2p-4)\ \dots\ 4x_3[K(X, Y)]^{-1},\\
D_{y_3}\Delta_Y^{p-1} G(X, Y)\big|_{y_3=0} &= 2(2p-2)(2p-3)(2p-4)\ \dots\ 2\cdot 1\cdot x_3[K(X, Y)]^{-3}.
\end{aligned}$$

It follows from these formulae that the function $u(X)$ assumes the form

$$u(X) = \sum_{i=0}^{p-1} U_i(X), \tag{5a}$$

where

$$U_i(X) = \frac{1}{4\pi}\iint_{E_2} f_i(y_1, y_2)x_3[K(X, Y)]^{2i-3}\, dy_1\, dy_2$$

for $i = 0, 1, 2, \dots, p-2, p-1$.

Let

$$W_1(X) = \{(x_1, x_2, x_3): x_i \in \langle A_i, B_i\rangle, \quad i = 1, 2, 3\}$$

where $A_i, B_i (i = 1, 2, 3)$ are arbitrary positive reals and $A_3 > 0$; let

$$Q_R = \{(y_1, y_2): y_1^2+y_2^2 > R^2\}.$$

By the triangle inequality we obtain for $R(W_1)$ sufficiently large

$$\frac{1}{4}(y_1^2+y_2^2) < y_1^2+y_2^2+(y_2-x_2)^2+x_3^2 < 4(y_1^2+y_2^2) \tag{6}$$

for each $X \in W_1$.

Lemma 1. Let the functions $f_i(y_1, y_2)$ be continuous and let the integrals

$$\iint_{E_2} |f_i(y_1, y_2)|(y_1^2+y_2^2)^{\frac{2i-3}{2}} dy_1 dy_2$$

be convergent for $i = 0, 1, \ldots, p-1$, then the integrals $U_i(X)$ $i = 0, 1, \ldots, p-1$ are almost uniformly convergent in the half-space $x_3 > 0$.

Proof. By Lemma 1 and (6) it follows that for every $\varepsilon > 0$ there exists a real number $R(\varepsilon, W_1)$ such that

$$\left|\iint_{Q_R} f_i(y_1, y_2)[K(X, Y)]^{2i-3} dy_1 dy_2\right| \leqslant \iint_{Q_R} |f_i(y_1, y_2)|(y_1^2+y_2^2)^{\frac{2i-3}{2}} dy_1 dy_2 < \varepsilon$$

for $X \in W_1$. This implies the conclusion of our Lemma.

Lemma 2. Let the functions $f_i(y_1, y_2)$ satisfy the assumptions of Lemma 1, then the function

$$\Delta_x^j U_i(X) \qquad i, j = 0, 1, \ldots, p-1$$

exists, and

$$\Delta_x^j U_i(X) = \frac{1}{4\pi} \iint_{E_2} f_i(y_1, y_2) \Delta_X^j [K(X, Y)]^{2i-3} dy_1 dy_2 . \tag{7}$$

Proof. It is enough to show that the integrals

$$\frac{1}{4\pi} \iint_{E_2} f_i(y_1, y_2) \Delta_X^j [K(X, Y)]^{2i-3} dy_1 dy_2$$

are uniformly convergent in W_1. Indeed

$$\Delta_X^j [K(X, Y)]^{2i-3} = P[(y_1-x_1)^{a_1} K^{2i-a_1-3},\ (y_2-x_2)^{a_2} K^{2i-a_2-3},\ x_3^{a_3} K^{2i-a_3-3}] ,$$

where $P(t_1, t_2, t_3)$ is a polynomial of a degree not exceeding 3^p; moreover

$$\left|\frac{y_1-x_1}{K}\right| < 1 , \qquad \left|\frac{y_2-x_2}{K}\right| < 1 , \qquad \left|\frac{x_3}{K}\right| < 1 .$$

Therefore for each positive ε there exists an $R = R(\varepsilon, W_1)$ such, that

$$\left|\iint_{Q_R} f_i(y_1, y_2) \Delta_X^j K^{2i-3}(X, Y) dy_1 dy_2\right| \leqslant C \iint_{Q_R} |f_i(y_1, y_2)|(y_1^2+y_2^2)^{\frac{2i-3}{2}} dy_1 dy_2 < \varepsilon ,$$

C being a constant, whence $\Delta_X^j U_i(X)$ exist and satisfy (7).

Lemma 3. Let the functions f_i satisfy the assumptions of Lemma 1, then the function $u(X)$ defined by (5a) satisfies the equation (1).

Proof. In virtue of (5a) and Lemma 2

$$\Delta_X^p u(X) = \sum_{i=0}^{p-1} \Delta_X^p U_i(X) = \frac{1}{4\pi} \sum_{i=0}^{p-1} \iint_{E_2} f_i(y_1, y_2) D_{y_3} \Delta_Y^{p-i-1} \Delta_X^p G(X, Y)\, dy_1 dy_2 .$$

The conclusion of our Lemma follows by symmetry in X and Y of the Green function and by Theorem 1.

Now we shall prove that the function $u(X)$ defined by (5) or (5a) satisfies the boundary conditions (2).

Lemma 4. Let the functions $f_i(y_1, y_2)$, $i = 0, 1, \ldots, p-2, p-1$ satisfy the assumption of Lemma 1 and let

$$\lim \Delta^j U_{i+j}(X) = 0 \quad \text{as} \quad X \to (x_1^0, x_2^0, 0+)$$

for $j, i = 0, 1, \ldots, p-1$, $i+j \leqslant p-1$, then the function $u(X)$ defined by (5) or (5a) satisfies the boundary conditions (2).

Proof. It was proved in paper [1] that

$$\lim U_0(X) = \lim \frac{1}{4\pi} \iint_{E_2} f_0(y_1, y_2) x_3 [K(X, Y)]^{-3} dy_1 dy_2 = f_0(x_1^0, x_2^0) ,$$

as $X \to (x_1^0, x_2^0, 0+)$. By Lemma 2

$$\Delta^i U_i(X) = \frac{1}{4\pi} \iint_{E_2} f_i(y_1, y_2) D_{y_3} \Delta_X^i K(X, Y)\, dy_1 dy_2 \to f_i(x_1^0, x_2^0) .$$

Lemmas 3 and 4 imply

Theorem 2. Let the assumptions of Lemmas 3 and 4 be satisfied, then the function $u(X)$ defined by (5) or (5a) solves the problem (1) and (2).

REFERENCES

[1] J. Musiałek, *Green's functions and the solutions of the Neumann and Dirichlet problems*, Commentationes Mathematicae XVI (1972), pp. 1—35.