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On Complete Linear, Metric, Conformal and Unimodular Classifications of Space of All-Finite Sequences of Vectors in a Given Vector Space

INTRODUCTION

Let a vector space V and natural numbers q and n satisfying the condition $n \leqslant \inf\{q, \dim V\}$ be given. Denoting $K = (1, \ldots, q)$ let us take an arbitrary n-element subsequence I of the sequence K. By $\underset{q}{X}_I$ let us denote a family of all q-element sequences $x = (x_k)$ $k \in K$ of vectors $x_k \in V$ such that the vectors (x_i) $i \in I$ are linearly independent and in the case $n < q$ the remaining vectors x_j $j \in J$, $J = K \setminus I$, may be written in the form

$$x_j = a_j^i x_i \qquad i \in I \qquad j \in J \qquad a_j^i \in R\,.$$

By $GL(V)$ we denote the group of all linear transformations of V. In [3] M. Kucharzewski gives the general solution of the functional equation

$$\varphi(Tx) = \varphi(x) \qquad x \in \underset{q}{X}_I \qquad T \in GL(V)$$

where $\varphi : \underset{q}{X}_I \to R$ is unknown, provided that $\dim V < \infty$.
In the case $n < q$ the result is the following: φ is a solution of the equation if there exists a function $\phi : R^n \times R^m \to R$, $m = q - n$, such that

$$\varphi(x) = \phi(a_j^i)\,.$$

In the case $n = q$ we take it that φ is a solution of the equation if it is a constant.

In the present paper we give a method of G-classification of space of all finite sequences of vectors in V for the following groups instead of G: linear, isometric, conformal, unimodular and, in the case dim $V < \infty$, also their subgroups consisting of transformations with positive determinants.

I. ON G-2-DECOMPOSITIONS

Suppose we are given an arbitrary group G of transformations of a set X onto itself. By G-orbit passing through a given point $x \in X$ we mean the set (see [1] [5])

$$\mathrm{Orb}_G(x) = \{x = T\underset{0}{x} : T \in G\} .$$

The space of all G-orbits in X we denote by $\mathrm{Orb}_G X$. Such a space gives a G-invariant decomposition of X.

Suppose we are given two decompositions [1]) of the set X, say $\{X_\alpha\}_{\alpha \in A}$ and $\{X_\beta\}_{\beta \in B}$; we say in brief that we are given a 2-decomposition of X.

Definition 1. (See [4]) A given 2-decomposition of X is said to be Cartesian if it satisfies the following condition

$$\forall_{\alpha \in A} \forall_{\beta \in B} \exists_{x \in X} : X_\alpha \cap X_\beta = \{x\} .$$

This notion may be generalized to n-decompositions.

Definition 2. A Cartesian 2-decomposition of X is said to be G-Cartesian if 1) it is G-invariant, 2) the components of one of the decompositions are G-orbits; the components of the second decomposition are called sections of the space of G-orbits or of X.

Let a G-Cartesian 2-decomposition be written in the form: $\{\mathrm{Orb}_G X, \mathrm{Sec}_G X\}$.

It may be asked whether G-Cartesian 2-decompositions of X exist for X and G given arbitrarily. The answer, in general, is not affirmative. But it makes sense to ask for a G-invariant decomposition $\{X_\lambda\}_{\lambda \in A}$ of X such that for each X_λ there exists a G-Cartesian 2-decomposition. So we may give.

Definition 3. By G-2-decomposition of X we mean a family $\{\mathrm{Orb}_G X_\lambda, \mathrm{Sec}_G X_\lambda\}_{\lambda \in A}$ of G-Cartesian 2-decompositions of X_λ such that $\{X_\lambda\}_{\lambda \in A}$ is a decomposition of X and each X_λ is maximal with respect to the property of G-Cartesian 2-decomposability; $A \subset X$, A being G-Cartesionally 2-decom-

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[1]) By a decomposition of a set X we mean a family of disjoint subsets of X the union of which over the family is equal to X.

posable, is said to be maximal if there does not exist a G-Cartesian 2-decomposition for any $B \subset X$ such that $B \supset A$ and $B \neq A$.

We shall prove the

Lemma 1. G-2-decompositions always exist.

Proof. Let us denote by Sub G the family of all subgroups of G and by $s\colon X \to \mathrm{Sub}\, G$ denote the mapping which maps $x \in X$ to the stationary (isotropy) subgroup G_x in G of x (see [5], [6]). The group G acts on $\mathrm{Sub}\, G$ by intrinsic automorphisms i.e $H' = THT^{-1}$ where $H, H' \in \mathrm{Sub}\, G$ and $T \in G$. Let us take an arbitrary section Σ of the space $\mathrm{Orb}_G\, \mathrm{Sub}\, G$ of G-orbits in Sub G, $\Sigma \subset \mathrm{Sub}\, G$, and find the coimage $\underset{0}{X} = s^{-1}(\Sigma)$ of Σ by the mapping s. It may be verified that for each $x \in X$ the condition $\mathrm{Orb}_G(x) \cap \underset{0}{X} \neq \emptyset$ is valid. So we may find a global section $\underset{0}{S}$ of the space $\mathrm{Orb}_G X$ of G-orbits in X such that $\underset{0}{S} \subset \underset{0}{X}$. If we write $\underset{0}{X} = \bigcup_{\sigma\in\Sigma} \underset{0}{X}_\sigma$, where $\underset{0}{X}_\sigma = s^{-1}(\sigma)$ and denote $\underset{0}{S}_\sigma = \underset{0}{S} \cap \underset{0}{X}_\sigma$, then it may be verified that for each $\sigma \in$ Range s the families $\{\mathrm{Orb}_G(x) : x \in \underset{0}{S}_\sigma\}$, $\{S_\sigma = T(\underset{0}{S}_\sigma) : T \in G\}$ form a G-Cartesian 2-decomposition of $X_\sigma = \bigcup_{x_0 \in \underset{0}{S}_\sigma} \mathrm{Orb}_G(x_0)$ and that each X_σ for $\sigma \in$ Range s is maximal with respect to the property of G-Cartesian 2-decomposability. The whole family forms a G-2-decomposition of X.

From the proof of Lemma 1 we directly obtain the two following lemmas.

Lemma 2. A set X is G-Cartesionally 2-decomposable if for each $\underset{1}{x}, \underset{2}{x} \in X$ their stationary subgroups $G_{\underset{1}{x}}$ and $G_{\underset{2}{x}}$ are conjugated.

Lemma 3. Points $\underset{1}{x}, \underset{2}{x} \in X$ belong to the same maximal subset of X if their stationary subgroups $G_{\underset{1}{x}}$ and $G_{\underset{2}{x}}$ are conjugated.

In the paper we construct certain G-2-decompositions of X for X being the space $\mathrm{Seq}\, V$ of all finite sequences of vectors in a given vector space V and for the following groups of transformations in V: $GL(V)$ — linear group, $GI(V)$ — isometric group, $GC(V)$ — conformal group and $GU(V)$ — unimodular group provided, of course, that for the last three groups V is given the structure of the inner product. Furthermore, in the case when $\dim V < \infty$ certain G-2-decompositions of $\mathrm{Seq}\, V$ are given also for the groups $GL^+(V)$, $GI^+(V)$, $GC^+(V)$ and $GU^+(V)$.

Let us notice that any section S of $\mathrm{Orb}_G X$ defines a G-invariant mapping $h : X \to S$ such that $h(\mathrm{Orb}_G(x)) = \mathrm{Orb}_G(x) \cap S$ for $x \in X$. If S is given a (local) chart $\xi : S \to R^n$ then $H_i = \xi_i \circ h$, $i = 1, \ldots, n$, form a complete and minimal system of G-invariants for the set h^{-1} (Domain ξ), which may be called a local complete system of G-invariants for X. The mapping h may be called a completely G-classifying mapping for X.

M. Kucharzewski in [2], [3] gives a method for the determination of local complete systems of $GL(V)$-invariants for $\mathrm{Seq}\, V$. In the present paper we propose a method for the determination of global complete systems of G-invariants for $\mathrm{Seq}\, V$ for every G listed.

1. $GL(V)$-2-decomposition of Seq$V$

In may easily be seen that Seq$V$ is not $GL(V)$-Cartesionally 2-decomposable. For the determination of a $GL(V)$-2-decomposition of Seq$V$ let us write $GL(V)$-invariant decomposition

$$\operatorname{Seq} V = \bigcup_{q \in N} \operatorname{Seq}(q) V \tag{1}$$

where $\operatorname{Seq}(q)V$, for fixed natural q, denotes the family of all sequences from Seq$V$ consisting of exactly q elements. In general, the sets $\operatorname{Seq}(q)V$ are also not $GL(V)$-Cartesionally 2-decomposable. But if for each q a $GL(V)$-2-decomposition of $\operatorname{Seq}(q)V$ be known, then a certain $GL(V)$-2-decomposition of Seq$V$ may easily be obtained from them; this will be shown later. For this reason we may fix q and consider the set $\underset{q}{X} = \operatorname{Seq}(q)V$ only. We shall determine a certain $GL(V)$-2-decomposition of this.

For this purpose we denote $K = (1, \ldots, q)$, $p = \min(q, \dim V)$ and introduce the family $K(p)$ of all subsequences of the sequence K consisting of at most p elements from K; we include in $K(p)$ also the empty set. Furthermore we introduce the mapping

$$\iota : \underset{q}{X} \to K(p) \tag{2}$$

defined in the following way:

Let $x = (x_k) \in X$. By $n(x)$ we denote the dimension of the subspace $V(x) \subset V$ generated by the system of vectors $x_1, \ldots, x_q$ and $\iota(x)$ we define as follows

1. If $x = 0$ we put $\iota(x) = \emptyset$.

2. If $x \neq 0$ then we put $\iota(x) = (i_1, \ldots, i_n)$ where $n = n(x)$ and $i_1, \ldots, i_n$ are obtained by using the following recurrent formula:

I. i_1 is the smallest element in K such that $x_{i_1} \neq 0$.

II. Suppose that $i_1, \ldots, i_\nu$ are determined and $\nu < n$. Then by $i_{\nu+1}$ we mean the smallest element in the sequence $i_\nu + 1, \ldots, q$ such that the vector $x_{i_{\nu+1}}$ is linearly independent of the system of vectors $x_{i_1}, \ldots, x_{i_\nu}$.

The mapping ι is a surjection.

Let us take an arbitrary $I \in K(p)$ and by $\underset{q}{X}_I$ let us denote the coimage of I by the mapping ι. Then we get the $GL(V)$-invariant decomposition

$$\operatorname{Seq}(q)V = \bigcup_{I \in K(p)} \underset{q}{X}_I . \tag{3}$$

We shall see that each $\underset{q}{X}_I$ is $GL(V)$-Cartesionally 2-decomposable but, in general, without the property of maximality in Seq$V$.

The cases $I = \emptyset$ and $I = K$ are trivial; the second case may be realized only for $q \leqslant \dim V$. In both cases the set $\underset{q}{X}_I$ is a $GL(V)$-orbit and $\operatorname{Sec}_{GL(V)} \underset{q}{X}_I$ is equal, by necessity, to the discrete decomposition of $\underset{q}{X}_I$.

If $I \neq \emptyset, K$ we denote $J = K \setminus I$ (subsequence of K). For an arbitrary $x \in \underset{q}{X}_I$ we may write $x = (x_i, x_j)$ $i \in I$ $j \in J$ and

$$x_j = a^i_j x_i \qquad j \in J \tag{4}$$

where a^i_j are uniquely determined and satisfy the conditions

$$j < i \Rightarrow a^i_j = 0\,. \tag{5}$$

By Mat (I, J) let us denote the family of all the matrices $A = \|a^i_j\|$ $i \in I$ $j \in J$ whose elements satisfy the condition (5).

Let E^L_n denote the family of all linearly independent systems of n vectors in V and let $n = |I|$. The group $GL(V)$ acts on E^L_n transitively.

By these denotations and $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_m)$ we formulate

Theorem 1. The mapping $\mathrm{Mat}(I, J) \times E^L_n \to \underset{q}{X}$ given by the formulas

$$\begin{cases} x_{i_\nu} = e_\nu \\ x_{j_\mu} a^{i_\nu}_{j_\mu} e_\nu \end{cases} \qquad \|a^i_j\| \in \mathrm{Mat}(I, J)(e_\nu) \in E^L_n \tag{6}$$

is a bijection and determines a $GL(V)$-2-decomposition of $\underset{q}{X}_I$ in such a way that the subspaces $\|a^i_j\| = \mathrm{const}$ are orbits and the subspaces $(e_\nu) = \mathrm{const}$ are sections.

We omit the proof because of its simplicity.

From the bijectivity of the mapping (6) it immediately follows that $\|a^i_j\|$ form a complete and minimal system of $GL(V)$-invariants for $\underset{q}{X}_I$.

If $I = \emptyset$ we get, instead of (6), the formulas

$$x_k = 0 \qquad k \in K \tag{6'}$$

and in the case $I = K$ — the formulas

$$x_k = e_k \qquad k \in K\,. \tag{6''}$$

For fixed n and q changed from n to infinity let us form the set

$$X_n = \bigcup_{q \geqslant n} \bigcup_{\substack{I \in K(p) \\ |I| = n}} \underset{q}{X}_I\,. \tag{7}$$

It may be said that the set X_n is a maximal subset of Seq$V$. For a given $(e_\nu) \in E^L_n$ the set

$$S_n(e_\nu) = \bigcup_{q \geqslant n} \bigcup_{\substack{I \in K(p) \\ |I| = n}} \underset{q}{S}_I(e_\nu)$$

where $\underset{q}{S}_I(e_\nu)$ denotes the section of $\underset{q}{X}_I$ determined by (e_ν) by means of formulas (6) or (6′), (6″), is a section of X_n. If (e_ν) varies in E^L_n, then sections S_n form a family $\mathrm{Sec}_{GL(V)} X_n$, $GL(V)$-invariant and having the property that together with $\mathrm{Orb}_{GL(V)} X_n$ it gives a $GL(V)$-Cartesian 2-decomposition of X_n.

If we denote

$$X_0 = \bigcup_{q \in N} \underset{q}{X}_\emptyset$$

then we get the decomposition

$$\operatorname{Seq} V = \bigcup_{n=0}^{\dim V} X_n$$

of the space $\operatorname{Seq} V$ to its maximal subspaces.

Let us put

$$S_0 = \bigcup_{q=1}^{\infty} \underset{q}{X}_\emptyset$$

and for each $n \geqslant 1$ let us choose arbitrarily one element (e_ν) in E_n^L and form the set

$$S = S_0 \cup \bigcup_{n=1}^{\dim V} S_n(e_\nu)\,, \tag{10}$$

where S_n are given by (8).

It may be said that the set S given by (10) is a (global) section of $\operatorname{Orb}_{GL(V)} \operatorname{Seq} V$. It defines a global complete and minimal system of $GL(V)$-invariants for $\operatorname{Seq} V$.

1^+. $GL^+(V)$-2-decomposition of Seq V

We assume that $\dim V < \infty$. To get a $GL^+(V)$-2-decomposition of $\operatorname{Seq} V$ we make use of the obtained $GL(V)$-2-decomposition of $\operatorname{Seq} V$. We consider $\underset{q}{X}_I$-subsets and distinguish the two cases:

a) $|I| < \dim V$, b) $|I| = \dim V$.

In the case a) the $GL(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$ given by the formulas (6) or (6′), (6″) is a $GL^+(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$.

In the case b) such a statement is not true, as follows from the fact that $GL(V)$ does not act on $E_n^L (n = \dim V)$ transitively. The space $\operatorname{Orb}_{GL^+(V)} E_n^L$ consists of two elements. Let us choose one of these denoted by $\overset{+}{E}{}_n^L$.

If we denote by M_n^ε a family of two matrices $\|\varepsilon_\nu^{\bar\nu}\|$ $\nu, \bar\nu = 1, \ldots, n$ equal to $\operatorname{Diag}[1, \ldots, 1, 1]$ or $\operatorname{Diag}[1, \ldots 1, -1]$, then we have Lemma 4. The mapping $\overset{+}{E}{}_n^L \times M_n \to E_n^L$ given by the formulas

$$e_\nu = \varepsilon_\nu^{\bar\nu} \bar e_{\bar\nu} \qquad \|\varepsilon_\nu^{\bar\nu}\| \in M_n^\varepsilon \qquad (\bar e_{\bar\nu}) \in E_n^L \tag{11}$$

is a bijection and gives a $GL(V)$-Cartesian 2-decomposition of E_n^L.

Now, using Lemma 4 and Theorem 1, we get

Theorem 1⁺. If $I \neq K$ and $|I| = \dim V$, then the mapping $\mathrm{Mat}(I, J) \times \times M_n^{\varepsilon} \to \overset{+}{E}{}_n^L \to \underset{q}{X}_I$ given by the formulas

$$(12) \qquad \begin{cases} x_{i_\nu} = \varepsilon_\nu^{\bar\nu} \bar e_{\bar\nu} \\ x_{j_\mu} = a_{j_\mu}^{i_\nu} \varepsilon_\nu^{\bar\nu} \bar e_{\bar\nu} \end{cases} \qquad \|a_j^i\| \in \mathrm{Mat}(I, J) \, \|\varepsilon_\nu^{\bar\nu}\| \in M_n^{\varepsilon} \ (\bar e_{\bar\nu}) \in \overset{+}{E}{}_n^L$$

is a bijection and determines a $GL(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$ in such a way that if we put $\|a_j^i\| = \mathrm{const}$, $\|\varepsilon_\nu^{\bar\nu}\| = \mathrm{const}$, then we get orbits and putting $(\bar e_{\bar\nu}) = \mathrm{const}$ we get sections.

From the bijectivity of the mapping (12) it immediately follows that $\|a_j^i\|$, $\|\varepsilon_\nu^{\bar\nu}\|$ form a complete and minimal system of $GL^+(V)$-invariants for $\underset{q}{X}_I$.

2. *GI(V)*-2-decomposition of Seq *V*

As before, we first consider $\underset{q}{X}_I$-sets. Each of these sets is $GI(V)$-Cartesionally 2-decomposable, as will be shown.

From the fact that $GI(V)$ is a subgroup of $GL(V)$ we infer by virtue of Theorem 1, that the problem of the determination of $GI(V)$-Cartesian 2-decomposition $\underset{q}{X}_I$ is reduced to the same problem for E_n^L instead of $\underset{q}{X}_I$.

Let us denote by M_n^I the family of all matrices $\|a_\nu^{\bar\nu}\|$, $\bar\nu, \nu = 1, \dots, n$ such that 1) $< \nu < \bar\nu \Rightarrow a_\nu^{\bar\nu} = 0$, 2) $a_\nu^\nu > 0$ for all $\nu, \bar\nu$ and by E_n^I the family of all orthonormal systems of n vectors in V.

Lemma 5. The mapping $M_n^I \times E_n^I \to E_n^L$ given by the formulas

$$(13) \qquad e_\nu = a_\nu^{\bar\nu} \bar e_{\bar\nu} \qquad \|a_\nu^{\bar\nu}\| \in M_n^I \qquad (\bar e_{\bar\nu}) \in E_n^I$$

is a bijection and gives a $GI(V)$-Cartesian 2-decomposition of E_n^L.

Proof. We shall prove only the bijectivity. The injectivity may be proved directly. For surjectivity we define a projection $f : E_n^L \to E_n^L$ by using the following recurrent formula: Let $(e'_\nu) = f(e_\nu))$

$$\text{I.} \qquad e'_1 = e_1 / |e_1| \, .$$

II. Supposing that $e'_1, \dots, e'_\nu$ are determined and $\nu < n$, we mean by $e'_{\nu+1}$ the vector uniquely determined by the following conditions

1) $|e'_{\nu+1}| = 1$
2) $e'_{\nu+1} \perp e_1, \dots, e_\nu$
3) $e'_{\nu+1} = \beta'_{\nu+1} e_1 + \dots + \beta_{\nu+1}^{\nu} e_\nu + \beta_{\nu+1}^{\nu+1} e_{\nu+1}$
4) $\beta_{\nu+1}^{\nu+1} > 0$

where $|e|$ denotes the norm of the vector e.

In consequence we get

$$e'_\nu = \beta_\nu^{\bar\nu} e_{\bar\nu} \tag{14}$$

where $(e'_\nu) \in E_n^I$ and $\|\beta_\nu^{\bar\nu}\| \in M_n^I$.

The mapping f is well-known as Schwartz's method of orthonormalization of bases in a vector space.

If we write the inverse of (14), then we get (13).

From Lemma 5 and Theorem 1 follows

Theorem 2. If $I \neq \emptyset$, K, then the mapping $\mathrm{Mat}(I,J)\times M_n^I \times E_n^I \to \underset{q}{X}_I$ where $n = |I|$, given by the formulas

$$\begin{cases} x_{i_\nu} = \alpha_\nu^{\bar\nu} e_{\bar\nu} \\ x_{j_\mu} a_{j_\mu}^{i_\nu} \alpha_\nu^{\bar\nu} e_{\bar\nu} \end{cases} \quad \|a_j^i\| \in \mathrm{Mat}(I,J)\, \|\alpha_\nu^{\bar\nu}\| \in M_n^I \quad (e_\nu) \in E_n^I \tag{15}$$

is a bijection and gives a $GI(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$ in such a sense that the subspaces $\|a_j^i\| = \mathrm{const}$ $\|\alpha_\nu^{\bar\nu}\| = \mathrm{const}$ are orbits and the subspaces $(e_\nu) = \mathrm{const}$ are sections.

From the bijectivity of the mapping (15) it follows that $\|a_j^i\|$ $\|\alpha_\nu^{\bar\nu}\|$ form a complete and minimal system of $GI(V)$-invariants for $\underset{q}{X}_I$.

If $I = \emptyset$ we take (6′) instead of (15) and in the case $I = K$ (15) is taken for (13).

The maximal $GL(V)$-subspaces X_n in $\mathrm{Seq}\,V$ are also maximal $GI(V)$-subspaces. In a manner similar to S in (10), a global section of $\mathrm{Orb}_{GI(V)}\mathrm{Seq}\,V$ (so also a global complete and minimal system of $GI(V)$-invariants for $\mathrm{Seq}\,V$) may be obtained.

2^+. $GI^+(V)$-2-decomposition of Seq$V$

We assume that $\dim V < \infty$.

If $|I| < \dim V$ any $GI(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$ is a $GI^+(V)$-Cartesian 2-decomposition, so we may use, for example, the formulas (15).

If $|I| = \dim V$ we get for $GI^+(V)$ only two orbits in E_n^I. Denoting one of these by $\overset{+}{E}{}_n^I$ we may formulate

Theorem 2^+. If $I \neq K$, then the mapping $\mathrm{Mat}(I,J)\times M_n^I \times M_n^{\varepsilon} \times \overset{+}{E}{}_n^I \to \underset{q}{X}_I$ given by the formulas

$$\begin{cases} x_{i_\nu} = \alpha_\nu^{\bar\nu} \varepsilon_{\bar\nu}^{\bar{\bar\nu}} e_{\bar{\bar\nu}}^I \\ x_{j_\mu} = a_{j_\mu}^{i_\nu} \alpha_\nu^{\bar\nu} \varepsilon_{\bar\nu}^{\bar{\bar\nu}} e_{\bar{\bar\nu}}^I \end{cases} \quad \|a_j^i\| \in \mathrm{Mat}(I,J)\, \|\alpha_\nu^{\bar\nu}\| \in M_n^I\, \|\varepsilon_{\bar\nu}^{\bar{\bar\nu}}\| \in M_n \quad (e_\nu^I) \in \overset{+}{E}{}_n^I \tag{16}$$

is a bijection and determines a $GI^+(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$ in which we get the orbits by putting $\|a^i_j\| = \text{const}$ $\|a^{\bar\nu}_\nu\| = \text{const}$ $\|\varepsilon^{\bar{\bar\nu}}_{\bar\nu}\| = \text{const}$ and the sections by putting $(e^I_{\bar{\bar\nu}}) = \text{const}$.

In the case when $I = \emptyset$ we take (6′) instead of (16) and in the case when $I = K$ we take only the first part of the equations (16).

Without any difficulty, if necessary, results may be formulated for the $GL^+(V)$-2-decomposition of $\mathrm{Seq}\,V$, global section of $\mathrm{Orb}_{GI^+(V)}\mathrm{Seq}\,V$ and global complete system of $GI^+(V)$-invariants for $\mathrm{Seq}\,V$.

3. *GC(V)*-2-decomposition of Seq *V*

We follow the same method as in the case of the group $GI(V)$. Denoting by M^C_n the family of all matrices from M^I_n such that $a^1_1 = 1$ and by E^C_n the family of all orthogonal systems of n vectors $e_1, \ldots, e_n$ in V satisfying the conditions $|e_1| = \ldots = |e_n|$ and putting $n = |I|$ we may formulate

Theorem 3. If $I \neq K$, then the formulas

$$(17)\qquad \begin{cases} x_{i_\nu} = a^{\bar\nu}_\nu e^C_{\bar\nu} \\ x_{j_\mu} = a^{i_\nu}_{j_\mu} a^{\bar\nu}_\nu e^C_{\bar\nu} \end{cases} \quad \|a^i_j\| \in \mathrm{Mat}(I, J)\, \|a^{\bar\nu}_\nu\| \in M^C_n \quad (e^C_\nu) \in E^C_n$$

define in the same way as the corresponding formulas in Theorems 1 or 2) a $GC(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$.

3⁺. *GC⁺(V)*-2-decomposition of Seq *V*

Assuming $\dim V < \infty$ and $|I| = \dim V$ and denoting by $\overset{+}{E}{}^C_n$ one of the two $GC^+(V)$-orbits in E^C_n we formulate the following

Theorem 3⁺. If $I \neq K$, then the formulas

$$(18)\qquad \begin{cases} x_{i_\nu} = a^{\bar\nu}_\nu \varepsilon^{\bar{\bar\nu}}_{\bar\nu} e^C_{\bar{\bar\nu}} \\ x_{j_\mu} = a^{i_\nu}_{j_\mu} a^{\bar\nu}_\nu \varepsilon^{\bar{\bar\nu}}_{\bar\nu} e^C_{\bar{\bar\nu}} \end{cases} \quad \|a^i_j\| \in \mathrm{Mat}(I, J)\, \|a^{\bar\nu}_\nu\| \in M^C_n\, \|\varepsilon^{\bar{\bar\nu}}_{\bar\nu}\| \in M_n \quad (e^C_{\bar{\bar\nu}}) \in \overset{+}{E}{}^C_n$$

define a $GC^+(V)$-Cartesian 2-decomposition of $\underset{q}{X}_I$.

The cases $I = \emptyset$ and $I = K$ we understand in a similar manner as previously.