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CCCCIII PRACE MATEMATYCZNE, ZESZYT 17 1975

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On a Method of Determination of Morphisms for *C*-objects

INTRODUCTION

In the paper we introduce the notion of C-object (see def. 3), C — a given concrete category in the sense of [9] ¹). If there is given an Ens-object $(X \times A, f, X)$, Ens being the category of sets, then X is nothing but an A — object (with respect to f) in the sense of [11]. In the case of Top — object, Top denoting the category of topological spaces, X is an A-space and in the case of Man-object, Man — the category of manifolds, X is an A — manifold (see [11]).

The family of all C-objects of types $+1$ and -1, for an arbitrarily fixed C, forms a category denoted by C-ob. The main purpose of the present paper is to give a method for construction of morphisms of this category. It is shown that the problem of determination of morphisms of the category C-ob reduces to the same problem for its subcategory of all C — objects of type $+1$ (see Prop. 2).

In the case of $C =$ Ens, the problem mentioned was treated in [12] and now we are chiefly interested in cases $C =$ Top and $C =$ Man. We use here a method which is a certain modification and extension of that given in [12].

There are four chapters in the paper. In the first we introduce the category C-ob of C — objects. In the second, we give the notions of orbital and homogeneous orbital sections for the mapping $f: X \times A \to X$ (see [2], [5]), f appearing in the notion of C-object, and adduce some theorems concerning the determination of such sections. The third describes a method of contracting morphisms of the category Ens-ob and the fourth characterizes morphisms of

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¹) We write C in the form $(\Omega_C, <, >_C)$, Ω_C — the space of objects of the category C and $<, >_C$ is a sign for its classes.

the category Man-ob. More precisely, we give a method for the determination of morphisms for homogeneous Man — objects. Homogeneity is here understood in the sense that the stationary subgroups of two arbitrary points in X are conjugated.

The notion of C — objects includes the notion of geometric objects ([1], [7]). It seems that we might speak of geometric C — objects, specially of geometric Man — objects. They also form a category which may be denoted, for example, by gC-ob. The notion of morphism of the category gC-ob is strictly connected with the notion of the concomitant of a geometric object (see [3], [4], [6], [8], [10], [17], [18]). By the method presented in [12], we have determined in [13], [14], [15] the concomitants of certain types for some geometric objects. Concomitants are of great importance in problems of classification of geometric objects (see [3], [16], [17]).

In the paper we use the language of categories and, in particular, we apply the following equalities (concerning the category of Ens) (see [2])

1. Let $f \in \langle A, A_1\rangle_{\mathrm{Ens}}$, $g \in \langle B, B_1\rangle_{\mathrm{Ens}}$, $h \in \langle A_1, A_2\rangle_{\mathrm{Ens}}$, $k \in \langle B_1, B_2\rangle_{\mathrm{Ens}}$. Then

$$(h\times k)\circ(f\times g) = (h\circ f)\times(k\circ g)\,.$$

2. Let $f \in \langle A, B\rangle_{\mathrm{Ens}}$, $g \in \langle A, C\rangle_{\mathrm{Ens}}$, $h \in \langle B, B_1\rangle_{\mathrm{Ens}}$, $k \in \langle B, B_2\rangle_{\mathrm{Ens}}$ and $q \in \langle C, C_1\rangle_{\mathrm{Ens}}$. Then

$$(h, k)\circ f = (h\circ f, k\circ f)$$

and

$$(h\times q)\times(f, g) = (h\circ f, q\circ g)\,.$$

Furthermore, we shall also use the equality (since it is easy to verify)

3. If $f \in \langle X, U\rangle_{\mathrm{Ens}}$, $g \in \langle X, V\rangle_{\mathrm{Ens}}$ and $h \in \langle Y, W\rangle_{\mathrm{Ens}}$, then

$$(f, g)\times h = \big(p_1\circ(f\times 1_Y), g\times h\big)$$

where $p_1 \in \langle U\times Y, U\rangle_{\mathrm{Ens}}$ is the canonical projection and $\langle Y, Y\rangle_{\mathrm{Ens}} \ni 1_Y$ is the identity mapping on Y.

I. CATEGORY OF C-OBJECTS

Let an arbitrary concrete category $C = (\Omega_C, \langle\ ,\ \rangle_C)$ be given (see [9], p. 38); its morphisms are mappings and its composition is the composition of mappings.

Definition 1. Any mapping $\times : \Omega_C\times\Omega_C\to\Omega_C$ is called a product-operation in C if the following condition is satisfied

$$\xi \in \langle A, B\rangle_C,\ \eta \in \langle C, D\rangle_C \Rightarrow \xi\times\eta \in \langle A\times C, B\times D\rangle_C\,.$$

Any category Man of differentiable manifolds of arbitrary class $r \in \{-1, 0, 1, 2, \ldots,\}$ is a category with a product operation; (by Man of class

$r = -1$ we mean the category Ens, by Man of class $r = 0$ — the category Top of topological spaces). The product-operation is the product of manifolds (see [2]).

Let a group (A, a) be given with group-operation $a : A \times A \to A$, neutral element e_a and inverse-operation $-1_a : A \to A$.

For sets $A_1, \dots, A_n$ and k-element variation $(i_1, \dots, i_k)$ given arbitrarily, with elements in $(1, \dots, n)$, we denote by $p^n_{i_1}, \dots, i_k$ the mapping $p^n_{i_1}, \dots, i_k :$ $A_1 x \dots x A_n \to A_{i_1} \times \dots \times A_{i_k}$
defined by

$$(1) \qquad p^n_{i_1,\dots,i_k}(a_1, \dots, a_n) := (a_{i_1}, \dots, a_{i_k}) \text{ for all } a_i \in A_i (i = 1, \dots, n) .$$

Denoting by $e_A : A \to A$ the constant mapping defined by $e_A(a) := e_a$ for all $a \in A$ we may state the following:

If (A, a) forms a group, then the identities hold

$$(2) \qquad \begin{aligned} &1)\ a \circ (p^3_1, a \circ p^3_{23}) = a \circ (a \circ p^3_{12}, p^3_3) , \\ &2)\ a \circ (1_A, e_A) = a \circ (e_A, 1_A) = 1_A \\ &3)\ a \circ (1_A, -1_a) = a \circ (-1_a, 1_A) = e_A \end{aligned}$$

where $p^3_1, p^3_3, p^3_{23}, p^3_{12}$ are defined in (1) for $A_1 = A_2 = A_3 = A$. The inverse is also true: If, for arbitrary set A, we have mappings $a : A \times A \to A, e_A : A \to A$ (constant mapping with value denoted by e_a) and $-1_a : A \to A$ such that all conditions 1)—3) in (2) are satisfied, then (A, a) forms a group with group-operation a, neutral element e_a and inverse operation -1_a.

Remark 1. If we denote $a^{-1} := a \circ p^2_{21}$, where p^2_{21} is defined in (1) for $A_1 = A_2 = A$, then it may be stated that (A, a^{-1}) is a group with neutral element e_a and inverse operation -1_a. This group is called the opposite group to the group (A, a) (see [11]).

The proof of this very well-known fact, in the Ens-category language, is the following:

We have to show that the following identities hold:

$$(3) \qquad \begin{aligned} &1)\ a^{-1} \circ (p^3_1, a^{-1} \circ p^3_{23}) = a^{-1} \circ (a^{-1} \circ p^3_{12}, p^3_3) , \\ &2)\ a^{-1} \circ (1_A, e_A) = a^{-1} \circ (e_A, 1_A) = 1_A , \\ &3)\ a^{-1} \circ (1_A, -1_a) = a^{-1} \circ (-1_a, 1_A) = e_A . \end{aligned}$$

To get the first identity in (3) we write

$$\begin{aligned} a^{-1} \circ (p^3_1, a^{-1} \circ p^3_{23}) &= (a \circ p^2_{21}) \circ (p^3_1, a \circ p^2_{21} \circ p^3_{23}) \\ &= (a \circ p^2_{21}) \circ (p^3_1, a \circ p^3_{32}) = a \circ (a \circ p^3_{32}, p^3_1) \end{aligned}$$

and

$$\begin{aligned} a^{-1} \circ (a^{-1} \circ p^3_{12}, p^3_3) &= (a \circ p^2_{21}) \circ (a \circ p^2_{21} \circ p^3_{12}, p^3_3) \\ &= (a \circ p^2_{21}) \circ (a \circ p^3_{21}, p^3_3) = a \circ (p^3_3, a \circ p^3_{21}) . \end{aligned}$$

From 1) in (2) we get the identity

$$\alpha \circ (p^3_1, \alpha \circ p^3_{23}) \circ p^3_{321} = \alpha \circ (\alpha \circ p^3_{12}, p^3_3) \circ p^3_{321}$$

or

$$\alpha \circ (p^3_3, \alpha \circ p^3_{21}) = \alpha \circ (\alpha \circ p^3_{32}, p^3_1)$$

which ends the proof of 1) in (3).
For the second identity in (3) we write

$$\alpha^{-1} \circ (1_A, e_A) = (\alpha \circ p^2_{21}) \circ (1_A, e_A) = \alpha \circ (e_A, 1_A) \underset{(2)}{=} 1_A$$

and

$$\alpha^{-1} \circ (e_A, 1_A) = (\alpha \circ p^2_{21}) \circ (e_A, 1_A) = \alpha \circ (1_A, e_A) \underset{(2)}{=} 1_A .$$

the third identity is obtained by way of

$$\alpha^{-1} \circ (1_A, -1_\alpha) = \alpha \circ p^2_{21} \circ (1_A, -1_\alpha) = \alpha \circ (-1_\alpha, 1_A) \underset{(2)}{=} e_A$$

and

$$\alpha^{-1} \circ (-1_\alpha, 1_A) = \alpha \circ p^2_{21} \circ (-1_\alpha, 1_A) = \alpha \circ (1_A, -1_\alpha) \underset{(2)}{=} e_A .$$

Denoting $1_\alpha := 1_A$ we may state the following.

Remark 2. For arbitrary $\varepsilon, \varepsilon' \in \{-1, 1\}$ we get the following identities

$$\begin{array}{ll} & 1.\ \alpha^\varepsilon = \varepsilon_\alpha \circ \alpha \circ (\varepsilon_\alpha \times \varepsilon_\alpha),\ (\varepsilon\varepsilon')_\alpha \circ e_A = e_A \\ (4) & 2.\ (\varepsilon\varepsilon')_\alpha = \varepsilon_\alpha \circ \varepsilon'_\alpha,\ (\varepsilon'_\alpha \times \varepsilon'_\alpha) \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big) = \varepsilon_\alpha \times \varepsilon_\alpha , \\ & 3.\ \varepsilon_\alpha \circ \varepsilon_\alpha = 1_\alpha \quad (\varepsilon\varepsilon')_\alpha \circ \varepsilon_\alpha = \varepsilon'_\alpha . \end{array}$$

The first identity, for $\varepsilon = -1$, is a direct consequence of the identity

$$-1_\alpha \circ \alpha = \alpha \circ (-1_\alpha \times -1_\alpha) \circ p^2_{21} ,$$

the rest are trivial.

Let C be a category with a product-operation "$\times$" such that for all $A \in \Omega_C$ we have $p^2_{21} \in \langle A \times A, A \times A \rangle_C$.

Definition 2. By a C-group we mean any group (A, α) such that

$$\begin{array}{ll} & 1.\ A \in \Omega_C , \\ (5) & 2.\ \alpha \in \langle A \times A, A \rangle_C , \\ & 3.\ -1_\alpha \in \langle A, A \rangle_C . \end{array}$$

Any C-group for $C =$ Man of an arbitrary class r is a Lie group of class r (see [10]).

Remark 3. The family of all C-groups forms a category denoted by C-gr; for given C-groups (A, α) and (B, β) the class $\langle (A, \alpha), (B, \beta) \rangle_{C\text{-gr}}$ consists (by definition) of all the mappings $\lambda : A \to B$ which have the following properties

$$\begin{array}{ll} (6) & 1.\ \lambda \in \langle A, B \rangle_C , \\ & 2.\ \lambda \circ \alpha = \beta \circ (\lambda \times \lambda) . \end{array}$$

The composition of mappings has the property of being a composition of morphisms of the type (6). Really, let

$$\lambda \in \langle (A, \alpha), (B, \beta)\rangle_{C\text{-gr}} \quad \text{and} \quad \mu \in \langle (B, \beta), (C, \gamma)\rangle_{C\text{-gr}}\,.$$

We shall show that $\mu \circ \lambda \in \langle (A, \alpha), (C, \gamma)\rangle_{C\text{-gr}}$. For this purpose let us write the conditions (6) for μ

$$\begin{aligned} &1.\ \mu \in \langle B, C\rangle_C\,,\\ &2.\ \mu \circ \beta = \gamma \circ (\mu \times \mu)\,. \end{aligned} \tag{7}$$

The fact that $\mu \circ \lambda \in \langle A, C\rangle_C$ immediately follows from 1) in (6) and in (7). It remains only to show the equality

$$(\mu \circ \lambda) \circ \alpha = \gamma \circ \big((\mu \circ \lambda) \times (\mu \times \lambda)\big)\,. \tag{8}$$

For the equality

$$(\mu \circ \lambda) \times (\mu \circ \lambda) = (\mu \times \mu) \circ (\lambda \times \lambda)$$

we may write

$$\begin{aligned} \gamma \circ \big((\mu \circ \lambda) \times (\mu \circ \lambda)\big) &= \gamma \circ \big((\mu \times \mu) \circ (\lambda \times \lambda)\big)\\ &\underset{(7)}{=} (\mu \circ \beta) \circ (\lambda \times \lambda) \underset{(6)}{=} \mu \circ (\lambda \circ \alpha) = (\mu \circ \lambda) \circ \alpha \end{aligned}$$

which was to be proved.

Remark 4. The opposite group (A, α^{-1}) of an arbitrary C-group (A, α) is also a C-group. Furthermore, for each $\varepsilon \in \{-1, 1\}$ we get

$$\alpha^{\varepsilon} \circ (1_A, -1_\alpha) = \alpha^{\varepsilon} \circ (-1_\alpha, 1_A) = e_A\,. \tag{8'}$$

Proof. We have $\alpha \in \langle A \times A, A\rangle_C$, $p^2_{21} \in \langle A \times A, A \times A\rangle_C$. Assuming that C is a concrete category we get $\alpha^{-1} \in \langle A \times A, A\rangle_C$ and this ends the proof (see (5)).

Definition 3. A triple $(X \times A, f, X)$ is called a $C(\alpha, f)$-object (or shortly C-object) of type ε, $\varepsilon \in \{-1, 1\}$, if

1) $X, A \in \Omega_C$,
2) $f \in \langle X \times A, X\rangle_C$,
3) $(A, \alpha) \in \Omega_{C\text{-gr}}$,
4) $f \circ (f \times 1_A) = f \circ (1_X \times \alpha^{\varepsilon})$,
5) $f \circ (1_X \times e_A) = p^2_1 \circ (1_X \times e_A)$

where $p^2_1 : X \times A \to X$ is defined in (1).

Remark 4. From Definition 3 we get the following fact. If $(X \times A, f, X)$ is a C-object of type ε, then

$$f \circ (f \times 1_A) \circ \big(1_X \times (1_A, -1_\alpha)\big) = p^2_1 \circ (1_X \times 1_A)\,. \tag{8''}$$

Proof. We write

$$\begin{aligned} f\circ(f\circ 1_A)\circ\big(1_X\times(1_A,-1_a)\big) &\underset{(4)}{=} f\circ(1_X\times\alpha^{\varepsilon})\circ\big(1_X\times(1_A,-1_a)\big)\\ &= f\circ\Big((1_X\circ 1_X)\times\big(\alpha^{\varepsilon 0}\circ(1_A,-1_a)\big)\Big)\\ &= f\circ\big(1_X\times\alpha^{\varepsilon 0}\circ(1_A,-1_a)\big)\\ &\underset{(8')}{=} f\circ(1_X\times e_A)\underset{(5)}{=} p_1^2\circ(1_X\times e_A)\\ &= p_1^2\circ(1_X\times 1_A)\,, \end{aligned}$$

which this was to be proved.
From (8′) we also get

$$(8''')\qquad f\circ(f\times 1_A)\circ\big(1_X\times(-1_a,1_A)\big)=p_1^2\circ(1_X\times 1_A)\,.$$

Definition 4. By a homomorphism of a given C-object $(X\times A, f, X)$ of type ε into another C-object $(Y\times B, g, Y)$ of type σ we mean any mapping

$$\varphi\times\lambda: X\times A\to Y\times B$$

such that

1. $\varphi\in\langle X, Y\rangle_C$,
2. $\lambda\in\langle(A,\alpha^{\varepsilon}),(B,\beta^{\sigma})\rangle_{C\text{-gr}}$,
3. $\varphi\circ f=g\circ(\varphi\times\lambda)$.

Proposition 1. The family of all C-objects forms a category in which morphisms are homomorphisms in the sense of definition 4 and the composition of morphisms is the composition of mappings.

Proof. Let be given three arbitrary C-objects

$$(X\times A, f, X),\ (Y\times B, g, Y),\ (Z\times C, h, Z)$$

of types ε, σ, ϱ, respectively, and arbitrary homomorphisms

$$\varphi\times\lambda: X\times A\to Y\times B,\ \psi\times\mu: Y\times B\to Z\times C\,.$$

We have to show that

$$(\psi\times\mu)\circ(\varphi\times\lambda): X\times A\to Z\times C$$

is a homomorphism.

Denoting $\chi:=\psi\circ\varphi$ and $\nu:=\mu\circ\lambda$ we may write

$$(\psi\times\mu)\circ(\varphi\times\lambda)=\chi\circ\nu\,.$$

By definition 4 we have to show that

1. $\chi\in\langle X, Z\rangle_C$,
2. $\nu\in\langle(A,\alpha^{\varepsilon}),(Z,\gamma^{\varrho})\rangle_{B\cdot\text{gr}}$
3. $\chi\circ f=h\circ(\chi\times\nu)$.

The first condition does not require any proof.

For the second let us write (6) and (7) for our morphisms

$$\lambda \in \langle (A, \alpha^{\varepsilon}), (B, \beta^{\sigma}) \rangle_{C\text{-gr}} \quad \text{and} \quad \mu \in \langle (B, \beta^{\sigma}), (C, \gamma^{\varrho}) \rangle_{C\text{-gr}} .$$

We have

$$\lambda \in \langle A, B \rangle_C , \quad \mu \in \langle B, C \rangle_C \tag{9}$$

and

$$\lambda \circ \alpha^{\varepsilon} = \beta^{\sigma} \circ (\lambda \times \lambda) , \quad \mu \circ \beta^{\sigma} = \gamma^{\varrho} \circ (\mu \times \mu) . \tag{10}$$

We have to show that (remember that $\nu := \mu \circ \lambda$)

$$\nu \in \langle A, C \rangle_C \tag{11}$$

and

$$\nu \circ \alpha^{\varepsilon} = \gamma^{\varrho} \circ (\nu \times \nu) . \tag{12}$$

The condition (11) is a direct consequence of (9); (12) we get in the following manner:

$$\gamma^{\varrho} \circ (\nu \times \nu) = \gamma^{\varrho} \circ \big((\mu \circ \lambda) \times (\mu \circ \lambda)\big) = \gamma^{\varrho} \circ \big((\mu \times \mu) \circ (\lambda \times \lambda)\big)$$
$$\underset{(10)}{=} (\mu \circ \beta^{\sigma}) \circ (\lambda \times \lambda) \underset{(10)}{=} \mu \circ (\lambda \circ \alpha^{\varepsilon}) = \nu \circ \alpha^{\varepsilon} .$$

For the third condition we have

$$\varphi \circ f = g \circ (\varphi \times \lambda) , \ \psi \circ g = h \circ (\psi \times \mu) \tag{13}$$

and we have to show that (remember: $\chi := \psi \circ \varphi, \nu := \mu \circ \lambda$)

$$\chi \circ f = h \circ (\chi \times \nu) .$$

We may write

$$h \circ (\chi \times \nu) = h \circ (\psi \circ \varphi) \times (\mu \circ \lambda) = h \circ (\psi \times \mu) \circ (\varphi \times \lambda)$$
$$\underset{(13)}{=} (\psi \circ g) \circ (\varphi \times \lambda) \underset{(13)}{=} \psi \circ (\varphi \circ f) = \chi \circ f .$$

The proposition 1 is therefore proved.

The category of C-objects we denote by C-ob.

The main purpose of our paper is to give a method for the determination of classes of morphisms of this category C-ob. First we shall show that the problem of the determination of the class of morphisms for a given pair of C-objects of types ε and σ, respectively, may be reduced to such a problem for C-objects of type 1. This fact follows directly from the following (putting $\varepsilon' = 1$ and $\sigma' = 1$).

Proposition 2. Suppose we are given two C-objects $(X \times A, f, X)$ and $(Y \times B, g, Y)$ of types ε and σ, respectively, and two numbers $\varepsilon', \sigma' \in \{-1, 1\}$. Denoting $1_{\alpha} := 1_A$ and

$$\bar{f} := f \circ \big(1_X \times (\varepsilon\varepsilon')_{\alpha}\big), \quad \bar{g} := g \circ \big(1_Y \times (\sigma\sigma')_{\beta}\big), \quad \bar{\lambda} := (\sigma\sigma')_{\beta} \circ \lambda \circ (\varepsilon\varepsilon')_{\alpha} \tag{14}$$

we may state the following

I) $(X\times A, \bar{f}, X)$ and $(Y\times B, \bar{g}, Y)$ are C-objects of types ε' and σ', respectively, and

II) $\varphi\times\lambda \in \langle(X\times A, f, X), (Y\times B, g, Y)\rangle_{C\text{-ob}} \Leftrightarrow$
$\varphi\times\bar{\lambda} \in \langle(X\times A, \bar{f}, X), (Y\times B, \bar{g}, Y)\rangle_{C\text{-ob}}$.

Proof of I). It is enough to show that $(X\times A, \bar{f}, X)$ is a C-object.

According to definition 3 we have to show that all conditions 1)—5) for our triple are satisfied. Conditions 1), 2) and 3) are satisfied trivially. The fourth condition takes the form

$$\bar{f}\circ(\bar{f}\times 1_A) = \bar{f}\circ(1_X\times\alpha^{\varepsilon'})\,. \tag{15}$$

We may write

$$\begin{aligned}\bar{f}\circ(\bar{f}\times 1_A) &\underset{(14)}{=} \big(f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\big)\circ\big(\big(f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\big)\times 1_A\big)\\ &= f\circ\big[\big(1_X\times(\varepsilon\varepsilon')_a\big)\circ\big(\big(f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\big)\times 1_A\big)\big]\\ &= f\circ\big[\big(1_X\circ\big(f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\big)\big)\times\big((\varepsilon\varepsilon')_a\circ 1_A\big)\big]\\ &= f\circ\big[\big(f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\big)\times(\varepsilon\varepsilon')_a\big]\end{aligned}$$

and

$$\begin{aligned}\bar{f}\circ(1_X\times\alpha^{\varepsilon'}) &\underset{(13)}{=} f\circ\big(1_X\times(\varepsilon\varepsilon')_a\big)\circ(1_X\times\alpha^{\varepsilon'}) = f\circ\big((1_X\circ 1_X)\big)\times\big((\varepsilon\varepsilon')_a\circ\alpha^{\varepsilon'}\big)\\ &= f\circ\big(1_X\times\big((\varepsilon\varepsilon')_a\circ\alpha^{\varepsilon'}\big)\big)\,.\end{aligned} \tag{16}$$

In virtue of (4) we may write

$$\begin{aligned}(\varepsilon\varepsilon')_a\circ\alpha^{\varepsilon'} &= (\varepsilon_a\circ\varepsilon'_a)\circ\big(\varepsilon'_a\circ\alpha\circ(\varepsilon'_a\times\varepsilon'_a)\big)\\ &= \varepsilon_a\circ\alpha\circ(\varepsilon'_a\times\varepsilon'_a)\\ &= \varepsilon_a\circ\alpha\circ\big((\varepsilon_a\circ\varepsilon_a\circ\varepsilon'_a)\times(\varepsilon_a\circ\varepsilon_a\circ\varepsilon'_a)\big)\\ &= \varepsilon_a\circ\alpha\circ\big(\big(\varepsilon_a\circ(\varepsilon\varepsilon')_a\big)\times\big(\varepsilon_a\circ(\varepsilon\varepsilon')_a\big)\big)\\ &= \varepsilon_a\circ\alpha\circ(\varepsilon_a\times\varepsilon_a)\circ\big((\varepsilon\varepsilon')_a\times(\varepsilon\varepsilon')_a\big)\\ &= \alpha^{\varepsilon}\circ\big((\varepsilon\varepsilon')_a\times(\varepsilon\varepsilon')_a\big)\end{aligned} \tag{17}$$

and

$$\begin{aligned}1_X\times\big((\varepsilon\varepsilon')_a\circ\alpha^{\varepsilon'}\big) &= 1_X\times\big(\alpha^{\varepsilon}\circ\big((\varepsilon\varepsilon')_a\times(\varepsilon\varepsilon')_a\big)\big)\\ &= (1_X\circ 1_X)\times\big(\alpha^{\varepsilon}\circ\big((\varepsilon\varepsilon')_a\times(\varepsilon\varepsilon')_a\big)\big)\\ &= (1_X\times\alpha^{\varepsilon})\circ\big(1_X\times\big((\varepsilon\varepsilon')_a\times(\varepsilon\varepsilon')_a\big)\big)\end{aligned}$$

so we get (see (16)) that

$$\begin{aligned} f \circ \left(1_X \times \left((\varepsilon\varepsilon')_\alpha \circ \alpha^{\varepsilon'}\right)\right) &= f \circ (1_X \times \alpha^{\varepsilon}) \circ \left(1_X \times \left((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\right)\right) \\ &\underset{\text{def 3}}{=} f \circ (f \times 1_A) \circ \left(1_X \times \left((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\right)\right) \\ &= f \circ (f \times 1_A) \circ \left(\left(1_X \times (\varepsilon\varepsilon')_\alpha\right) \times (\varepsilon\varepsilon')_\alpha\right) \\ &= f \circ \left(f \circ \left(1_X \times (\varepsilon\varepsilon')_\alpha\right)\right) \times \left(1_A \circ (\varepsilon\varepsilon')_\alpha\right) \\ &= f \circ \left(f \circ \left(1_X \times (\varepsilon\varepsilon')_\alpha\right) \times (\varepsilon\varepsilon')_\alpha\right) . \end{aligned}$$

The identity (15) is therefore proved.

The fifth condition in def. 3 for $(X \times A, \bar{f}, X)$ follows from the identity (which is easy to verify)

$$\bar{f} \circ (1_X \times e_A) = f \circ (1_X \times e_A) .$$

Proof of II). We assume that (see def. 4)

$$(18) \qquad \begin{aligned} &1.\ \varphi \in \langle X, Y \rangle_C , \\ &2.\ \lambda \in \langle (A, \alpha^{\varepsilon}), (B, \beta^{\sigma}) \rangle_{C\text{-gr}} \\ &3.\ \varphi \circ f = g \circ (\varphi \times \lambda) \end{aligned}$$

and we have to show that (see (14))

$$(19) \qquad \begin{aligned} &1.\ \varphi \in \langle X, Y \rangle_C , \\ &2.\ \bar{\lambda} \in \langle (A, \alpha^{\varepsilon'}), (B, \beta^{\sigma'}) \rangle_{C\text{-gr}} , \\ &3.\ \varphi \circ \bar{f} = \bar{g} \circ (\varphi \times \bar{\lambda}) . \end{aligned}$$

The first conditions in (18) and (19) are the same, to get the second condition in (19) we have to show that (see (6))

$$(20) \qquad \begin{aligned} &1)\ \bar{\lambda} \in \langle A, B \rangle_C , \\ &2)\ \bar{\lambda} \circ \alpha^{\varepsilon'} = \beta^{\sigma'} \circ (\bar{\lambda} \times \bar{\lambda}) . \end{aligned}$$

The first condition in (20) does not require any proof.

For the second we write

$$\bar{\lambda} \circ \alpha^{\varepsilon'} \underset{(14)}{=} (\sigma\sigma')_\beta \circ \lambda \circ (\varepsilon\varepsilon')_\alpha \circ \alpha^{\varepsilon'} \underset{(17)}{=} (\sigma\sigma')_\beta \circ \lambda \circ \alpha^{\varepsilon} \circ \left((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\right)$$

and

$$\begin{aligned} \beta^{\sigma} \circ (\bar{\lambda} \times \bar{\lambda}) &\underset{(14)}{=} \beta^{\sigma'} \circ \left((\sigma\sigma')_\beta \circ \lambda \circ (\varepsilon\varepsilon')_\alpha\right) \times \left((\sigma\sigma')_\alpha \circ \lambda \circ (\varepsilon\varepsilon')_\alpha\right) \\ &= \beta^{\sigma'} \circ \left[\left((\sigma\sigma')_\beta \circ \lambda\right) \times \left((\sigma\sigma')_\beta \circ \lambda\right)\right] \circ \left[(\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\right] \\ &= \beta^{\sigma'} \circ \left((\sigma\sigma')_\beta \times (\sigma\sigma')_\beta\right) \circ (\lambda \times \lambda) \circ \left((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\right) \end{aligned}$$

$$
\begin{aligned}
&\underset{(4)}{=} \sigma'_\beta \circ \beta \circ (\sigma'_\beta \times \sigma'_\beta) \circ \big((\sigma\sigma')_\beta \times (\sigma\sigma')_\beta\big) \circ \lambda \times \lambda) \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big)\\
&\underset{(4)}{=} \sigma'_\beta \circ \beta \circ (\sigma_\beta \times \sigma_\beta) \circ (\lambda \times \lambda) \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big)\\
&\underset{(4)}{=} (\sigma\sigma')_\beta \circ \sigma_\beta \circ \beta \circ (\sigma_\beta \times \sigma_\beta) \circ (\lambda \times \lambda) \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big)\\
&\underset{(4)}{=} (\sigma\sigma')_\beta \circ \beta^\sigma \circ (\lambda \times \lambda) \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big)\\
&\underset{(6)}{=} (\sigma\sigma')_\beta \circ \lambda \circ \alpha^\varepsilon \circ \big((\varepsilon\varepsilon')_\alpha \times (\varepsilon\varepsilon')_\alpha\big)\,.
\end{aligned}
$$

Now we turn to the third condition in (19). We write

$$
\begin{aligned}
\varphi \circ \bar f &\underset{(14)}{=} \varphi \circ f \circ \big(1_X \times (\varepsilon\varepsilon')_\alpha\big) \underset{(4)}{=} \varphi \circ f \circ \big((1_X \circ 1_X) \times (\varepsilon_\alpha \circ \varepsilon'_\alpha)\big)\\
&= \varphi \circ f \circ (1_X \times \varepsilon_\alpha) \circ (1_X \times \varepsilon'_\alpha) \underset{(18)}{=} g \circ (\varphi \times \lambda) \circ (1_X \times \varepsilon_\alpha) \circ (1_X \times \varepsilon'_\alpha)\\
&= g \circ \big((\varphi \circ 1_X) \times (\lambda \circ \varepsilon_\alpha)\big) \circ (1_X \times \varepsilon'_\alpha) = g \circ \big(\varphi(\lambda \circ \varepsilon_\alpha)\big) \circ (1_X \times \varepsilon'_\alpha)\\
&= g \circ \big((\varphi \circ 1_X) \times (\lambda \circ \varepsilon_\alpha \circ \varepsilon'_\alpha)\big) \underset{(4)}{=} g \circ \Big(\varphi \times \big(\lambda \circ (\varepsilon\varepsilon')_\alpha\big)\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar g \circ (\varphi \times \bar\lambda) &\underset{(14)}{=} g \circ \big(1_Y \times (\sigma\sigma')_\beta\big) \circ \Big(\varphi \times \big((\sigma\sigma')_\beta \circ \lambda \circ (\varepsilon\varepsilon')_\alpha\big)\Big)\\
&= g \circ \Big((1_Y \circ \varphi) \times \big((\sigma\sigma')_\beta \circ (\sigma\sigma')_\beta \circ \lambda \circ (\varepsilon\varepsilon')_\alpha\big)\Big)\\
&\underset{(4)}{=} g \circ \Big((1_X \circ \varphi) \times \big(1_\beta \circ \lambda \circ (\varepsilon\varepsilon)_\alpha\big)\Big) = g \circ \Big(\varphi \times \big(\lambda \circ (\varepsilon\varepsilon')_\alpha\big)\Big)\,.
\end{aligned}
$$

Therefore "⇒" in II is proved. The proof of "⇒" is the same as for "⇒" because we have (see (14))

$$\overline{(\bar f)} = f,\ \overline{(\bar g)} = g,\ \overline{(\bar\lambda)} = \lambda\,.$$

Proposition 2 is proved.

II. ORBITAL SECTIONS OF MAN-OBJECTS

From now on we shall only consider C-objects of type 1. We shall now give some definitions.

Let a C-object $(X \times A, f, A)$ of type 1 be given and let us fix arbitrarily $x_0 \in X$.

Definition 5. By an orbit of x_0 we mean the set

$$\mathrm{Orb}\, x_0 := \{x \in X \mid \exists\, a \in A : x = f(x_0, a)\}\,.$$

Each two orbits are either identical or disjoint.

The space of all orbits in X we denote by $\mathrm{Orb}\, X$.

Definition 6. By stationary subgroup of x_0 (or isotropy group of x_0 or stabilizer of x_0) we mean the subgroup A_{x_0} of the group A defined by

$$A_{x_0} := \{a \in A \mid f(x_0, a) = x_0\}\,^{2)}\,. \tag{21}$$

Definition 6′. By point — stationary subgroup of a given set M in X (point-isotropy group of M) we mean the intersection of all stationary subgroups of points in M. We denote this by A_M.

Definition 6″. By set-stationary subgroup of M we mean the subgroup A_M of A defined by

$$A_M := \{a \in A \mid f(M, a) = M\}\,.$$

In this paper we shall apply A_M only in the sense of definition 6′. Definition 6″ is given only for comparison with the former.

Definition 7. A mapping $s : X \to X \times A$ is called a section for C-object $(X \times A, f, X)$ if

$$\begin{aligned} &1)\ f \circ s = 1_X\,;\\ &2)\ s \in \langle X, X \times A\rangle_C\,. \end{aligned} \tag{22}$$

Denote $p_1 := p_1^2$, $p_2 := p_2^2$ (see (1)).

Remark 5. If s is a section for C-object $(X \times A, f, X)$, then the mapping $p_1 \circ s : X \to X$ may be projected to an identity mapping $\operatorname{Orb} X \to \operatorname{Orb} X$; In other words, the decomposition of X given by orbits is invariant under the mapping $p_1 \circ s$.

Proof. Let us take an arbitrary $x \in X$. We have $(f \circ s)(x) = x$. Denote $(x', a') := s(x)$. From $f(x', a') = x$ it follows that $x' \in \operatorname{Orb} x$. But $x' = (p_1 \circ s)(x)$, so $(p_1 \circ s)(x) \in \operatorname{Orb} x$, which was to be shown.

Definition 8. A section s of C-object $(X \times A, f, X)$ is called orbital if the following identity holds true

$$p_1 \circ s \circ f = p_1 \circ \big((p_1 \circ s) \times 1_A\big)\,. \tag{23}$$

This condition is equivalent to the fact that $p_1 \circ s$ is constant on each orbit. Denote

$$\underset{0}{X} := Im(p_1 \circ s) \tag{24}$$

and for a given $x_0 \in \underset{0}{X}$ denote

$$\underset{x_0}{A} := Im(p_2 \circ s \mid \operatorname{Orb} x_0)\,. \tag{25}$$

Definition 9. For a given disjoint decomposition $\operatorname{Dec} X$ of a set X we call $\underset{0}{X} \subset X$ by a section of $\operatorname{Dec} X$ if $\underset{0}{X}$ meets any component of $\operatorname{Dec} X$ at exactly one point.

---

[2] From the property 5) of f assumed in def. 3 it follows that for each $x_0 \in X$ we have $A_{x_0} \neq \emptyset$.