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Remarks on J. Gancarzewicz's Paper "On Commutative Algebraic Objects over a Groupoid"

J. Gancarzewicz has introduced in [1] the notion of a commutative algebraic object and has studied its properties. Unfortunately in some proofs in this note incorrect statements have been used although these proofs can be established without them. Furthermore Theorem 3 and Lemma 3 should be slightly changed because in the form given in [1] they are not true. The purpose of this note is to point out the inaccuracies noticed in [1] and to signal how they can be corrected. All non-defined notions in this paper are used in the sense of the definitions given in [1] and [4].

In the proofs of Corollary 1, Lemma 4 and Theorem [5] in [1] the author has used the following statement (cf. [1] p. 19): "Let (A, X) and (B, X) be two algebraic objects over X, and let $h: A \to B$ be a strong homomorphism of (A, X) into (B, X). If (C, X) is a subobject of (A, X), then (C, X) is a subobject of (B, X)".

We shall prove that this is true if X is a Brandt groupoid but it is not true in a more general case if X is only a groupoid.

Theorem. Let (A, X) be a complete algebraic object, (B, X) an arbitrary algebraic object and let $h: A \to B$ be a homomorphism of (A, X) into (B, X). If (C, X) is a subobject of (A, X), then $\big(h(C), X\big)$ is a subobject of (B, X).

Proof. Let $x \in X$, $c \in C$. Then $xh(c)$ is defined (as xc is defined and h is a homomorphism) and

$$xh(c) = h(xc) .$$

But C is a stable subset (cf. [4] Proposition 3 p. 72) and hence $xc \in C$ and consequently $xh(c) \in h(C)$. Thus $h(C)$ is a stable subset i.e. $\big(h(C), X\big)$ is a subobject of (B, X).

In [2] it has been proved that every algebraic object over a Brandt groupoid is complete [1]) (cf. [2]. Corollary 1 p. 226). This fact and the previous theorem immediately imply the following.

Corollary. Let (A, X), (B, X) be two algebraic objects over a Brandt groupoid X and let $h: A \to B$ be a homomorphism of (A, X) into (B, X). If (C, X) is a subobject of (A, X), then $\big(h(C), X\big)$ is a subobject of (B, X).

We shall now show that the analogue of Corollary 1 for a groupoid is not valid.

Example 1. Let $X = X_1 \cup X_2$, where

$$X_1 = \{1, 2\}\times\{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\},$$

$$X_2 = \{3, 4\}\times\{3, 4\} = \{(3, 3), (3, 4), (4, 3), (4, 4)\}.$$

We define an interior operation "$\cdot$" in X in the following way:

$$(i, j)\cdot(k, l) = (i, l) \quad \text{for} \quad j = k,$$

$(i, j)\cdot(k, l)$ is non-defined for $j \neq k$.
It may be seen that X_1 and X_2 with such interior operation are Brandt groupoids and consequently X is a groupoid (cf. [3] p. 10).

Let $A = (-\infty, 0\rangle \cup \langle 1, \infty)$. We put:

$$(1, 1)a = (2, 2)a = a \quad \text{for} \quad a \in \langle 1, \infty),$$

$$(1, 2)a = e^{a-1} \quad \text{for} \quad a \in \langle 1, \infty),$$

$\left(e \text{ denotes the number } \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)$,

$$(2, 1)a = \ln a + 1 \quad \text{for} \quad a \in \langle 1, \infty),$$

$$(i, j)a = \frac{i}{j}a \quad \text{for } a \in (-\infty, 0\rangle, i, j = 3, 4.$$

It is easy to verify that the pair (A, X) with the exterior product defined in such a way is an algebraic object.
Let $B = \langle 0, \infty)$. We define the exterior product $(i, j)a$ for $(i, j) \in X_1$, $a \in \langle 1, \infty)$ in the same way as in (A, X), and we put $(i, j)a = \frac{i}{j}a$ for $a \in \langle 0, \infty)$, $i, j = 3, 4$.
It is seen that (B, X) is an algebraic object.
Let us consider the function $h: A \ni a \to |a|$. It can be easily verified that h is a strong homomorphism of (A, X) into (B, X). Set $\langle 1, \infty)$ is stable in (A, X) but it is not stable in (B, X) (e.g. $(3, 4)\frac{5}{4} = \frac{15}{16} \notin \langle 1, \infty)$). Thus $(\langle 1, \infty), X)$ is a subobject of (A, X), but $(h(1, \infty), X)$ is not a subobject of (B, X).

Now we shall give an example showing that Theorem 3 and Lemma 3 of [1] are incorrect.

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[1]) In [2] the term "a non-singular object" has been used instead of the term "a complete object".

Example 2. Let X_1 be the same groupoid as in Example 1. As a fibre we take set Z of integers. We put:

$$h_1(n) = n+1 \quad \text{for} \quad n \in Z,$$
$$h_2(n) = n \quad \text{for} \quad n \in Z,$$
$$(i,j)n = h_i^{-1}h_j(n) \quad \text{for} \quad n \in Z, \; i,j = 1,2.$$

The pair (Z, X_1) with an exterior product defined in such way is an algebraic object (cf. e.g. [2] Theorem 3 and Corollary 2 p. 222, 226). Only set Z actually is stable in this object. Suppose that a set $\tilde{Z}$, $\phi \neq \tilde{Z} \subset Z$ is stable. Let $n_0 \in \tilde{Z}$. Then we have:

$$(2,1)n_0 = n_0+1 \in \tilde{Z},$$

and consequently (by induction) we obtain:

$$n \in \tilde{Z} \quad \text{for} \quad n \geqslant n_0.$$

In a similar way we have:

$$(1,2)n_0 = n_0-1 \in \tilde{Z}$$

which consequently implies that $n \in \tilde{Z}$ for $n < n_0$. Thus $\tilde{Z} = Z$.
The object (Z, X) does not contain a proper subobject and hence it is simple. Obviously (Z, X) is non-transitive. This proves that Theorem 3 of [1] is incorrect. Every one-element set $\{n\}$, $n \in Z$ is a generator of (Z, X) (as (Z, X) is simple). But we have for $(i,j) \in X_1$:

$$(i,j)1 = 1 \quad \text{or} \quad (i,j)1 = 2 \quad \text{or} \quad (i,j)1 = 0.$$

This shows that Lemma 3 of [1] is not true. It also proves that the object $0(a)$ may be non-transitive (which has been used in the proofs of Theorems 3, 5, 6 of [1]).

All the cases of incorrectness of [1] shown have been caused by the fact that the "multiplication" in a groupoid is defined for only some pairs. Let us observe that all results of [1] (and all proofs) will be correct if we replace a semigroupoid by a semigroup with the unit a groupoid by a group. But the results of [1] can also be corrected without the loss of their generality, i.e. without additional assumptions. For this purpose we need a certain notion more general than a transitivity. We shall now define this notion.

Let (A, X) be an algebraic object. By $\tilde{X}$ we shall denote the set of all finite non-empty sequences of elements of X. The exterior product $X \times A \ni (x, a) \to xa$ can be extended to the exterior product $\tilde{X} \times A \ni (\tilde{x}, a) \to xa$. We put for $a \in A$, $(x_n, \ldots, x_1) \in \tilde{X}$:

$$(x_n, x_{n-1}, \ldots, x_2, x_1)a = x_n\Big(x_{n-1}(\ldots x_2(x_1, a)\ldots)\Big),$$

where the left side of this equality is defined iff the right side is defined.

We propose to accept the following

Definition. An algebraic object (A, X) over X will be called quasi-transitive if for every $a, b \in A$ there exists $\tilde{x} \in \tilde{X}$ such that $\tilde{x}a = b$.

It is immediately seen that the notion of quasi-transitive algebraic object is a generalization of the notion of a transitive algebraic object. On the other hand, if a semigroupoid X is a semigroup (obviously with a unit), then we have for every object (A, X):

$$(x_n, x_{n-1}, \dots, x_2, x_1)a = x_n\big(x_{n-1}(\dots x_2(x_1 a)\dots)\big) = (x_n \cdot x_{n-1} \cdot \dots \cdot x_2, x_1)a .$$

But $(x_n \cdot x_{n-1} \dots x_2 \cdot x_1) \in X$. Thus, if in the case considered an object (A, X) is quasi-transitive, then (A, X) is also transitive. This means that in the case when X is a semigroup with a unit, the notion of a quasi-transitive object is equivalent to the notion a transitive object.

If we replace in [1] the term "transitive" by the term "quasi-transitive" and "$x \in X$" by "$\tilde{x} \in \tilde{X}$" respectively, then all results of this note are true (though some proofs have to be changed).

This will be proved in detail in an other note, now being prepared. We must first investigate some properties of the quasi-transitive objects; this will be reported in the forthcoming paper.

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