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Analytic Continuation of Harmonic Functions

1. INTRODUCTION

In this paper by D we denote an arbitrary fixed, open, connected and not empty subset (a region) of $\boldsymbol{R}^n$, $n \geqslant 2$. It is known that every function h harmonic on D may be continued to a holomorphic function in an open set $\tilde{D}_h \subset \boldsymbol{C}^n$. It may be asked whether there exists an open connected set $\tilde{D} \subset \boldsymbol{C}^n$ such that $D \subset \tilde{D}$ and every harmonic function on D may be continued to a holomorphic (or only to an analytic multivalued) function on $\tilde{D}$. It may also be asked whether there exists a maximal set $\tilde{D}$ with these properties. This set will be called *a harmonic envelope of holomorphy* (or *of analyticity*) for D.

From the paper [2] we can deduce the following

Theorem I. For every region $D \subset \boldsymbol{R}^n$ there exists a harmonic envelope of analyticity.

In the paper [3] we can find

Theorem II. For every region $D \subset \boldsymbol{R}^n$ there exists an open connected set $\tilde{\tilde{D}} \subset \boldsymbol{C}^n$ such that every harmonic function on D may be continued to a holomorphic function on $\tilde{\tilde{D}}$.

Theorem III. If $B = \{x \in \boldsymbol{R}^n : |x| < r\}$, $n \geqslant 2$, $r > 0$, then

$$\tilde{B} = \{z = x+iy \in \boldsymbol{C}^n : [|x|^2 + |y|^2 + 2(|x|^2|y|^2 - \langle x, y\rangle^2)^{\frac{1}{2}}]^{\frac{1}{2}} < r\}$$

is the harmonic envelope of holomorphy for B.

In his paper [2] Lelong presented two methods of construction of a harmonic envelope of analyticity. In Section 2 of this paper these two methods are analysed and used for the effective construction of a harmonic envelope of analyticity for the ball and the spatial ring. By different methods the harmonic envelope of holomorphy of the ball was obtained in [1] and [3]. The paper is closed by Section 3 in which theorems 7 and 8 are proved. Theorem 7 permits the effective construction of the harmonic envelope of analyticity for $D \subset \boldsymbol{C}$, if a harmonic envelope of analyticity is known for some region that is biholomorphically equivalent to D. Theorem 8 permits the construction of a harmonic envelope of holomorphy for the set $D \subset \boldsymbol{C}$ that is biholomorphically equivalent to the unit disc.

Now we present a list of the denotations used in this note.

As usual for $A \subset \boldsymbol{C}^n$ (or $A \subset \boldsymbol{R}^n$) by $A^0, \bar{A}, \partial A$ we denote, respectively, the interior, the closure and the boundary of A. For $U = U^0 \subset \boldsymbol{R}^n$, by $H(U)$ we denote the set of all harmonic functions on U. For $\Omega = \Omega^0 \subset \boldsymbol{C}^n$, by $O(\Omega)$ we denote the set of all holomorphic functions on Ω. For arbitrary $z, w \in \boldsymbol{C}^n$, by $\langle z, w\rangle$ we denote the standard scalar product in $\boldsymbol{C}^n$ (i.e. $\langle z, w\rangle = \sum_{j=1}^{n} z_j \overline{w}_j$) and by $|z|$ the norm induced by the scalar product.

2. LELONG SETS

The whole of this section has been suggested by the ideas contained in Lelong's paper [2]. It has the character of a short report on results relative to the analytic continuation of harmonic functions.

The first part of this section is devoted to generalizations of Lelong's methods of construction of a harmonic envelope of analyticity. The case $n = 2$ plays a special role in this theory, therefore we shall devote most attention to this.

Now, we shall define two sets, which play a fundamental role in the following constructions.

Let $F(z) = \sum_{j=1}^{n} z_j^2$, $z = (z_1, \ldots, z_n) \in \boldsymbol{C}^n$. For $z_0 \in \boldsymbol{C}^n$, $t_0 \in \boldsymbol{R}^n$ let $T(z_0) = \{t \in \boldsymbol{R}^n : F(z_0 - t) = 0\}$, $\Gamma(t_0) = \{z \in \boldsymbol{C}^n : F(z - t_0) = 0\}$.

From these definitions we can directly obtain the following

Lemma 1.

(a) For $z = x + iy \in \boldsymbol{C}^n : T(z) = \{t \in \boldsymbol{R}^n : |x - t| = |y|, \langle x = t, y\rangle = 0\}$;

(b) for $z = x + iy \in \boldsymbol{C}^n : T(z) = T(\bar{z}), T(x) = \{x\} / \bar{z} = x - iy /$;

(c) in the case $n \geqslant 2$, for $z = x + iy \in \boldsymbol{C}^n$, $y \neq 0$ the set $T(z)$ is an $(n-2)$ dimensional sphere with the center x, the radius $|y|$ and $T(z)$ lies in the hyperplane $\{t \in \boldsymbol{R}^n : \langle x - t, y\rangle = 0\}$;

(d) in the case $n = 2$, for $z = (z_1, z_2) \in \boldsymbol{C}^2$: $T(z) = \{z + iz_2, \bar{z}_1 + i\bar{z}_2\}$ (we identify $\boldsymbol{C}$ and $\boldsymbol{R}^2$);

(e) in the case $n \geqslant 3$ the set $T(z)$ is connected;

(f) for $t \in \boldsymbol{R}^n$ $\Gamma(t) \cap \boldsymbol{R}^n = \{t\}, (n = 1 : \Gamma(t) = \{t\})$;