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On a Certain Polyharmonic Problem for the Half-Plane

In the present paper we shall give a solution of the m -harmonic equation

$$(1) \quad \Delta^m u(x_1, x_2) = 0$$

in the half-plane $x_2 > 0$ satisfying the boundary conditions

$$(2) \quad \Delta^i (D_{x_2} + h)u(x_1, 0) = f_i(x_1), \quad \Delta^i = \Delta(\Delta^{i-1}),$$

where $i = 0, 1, \dots, m-1$ and h is a negative constant.

We shall solve the problem (1), (2) by means of the convenient Green function.

1. Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ denote arbitrary points for which $x_2 > 0, y_2 \geq 0$ and let $r_1^2 = |XY|^2 = (y_1 - x_1)^2 + (y_2 - x_2)^2, r_2^2 = |X_1 Y|^2 = (y_1 - x_1)^2 + (y_2 + x_2)^2$, where $X_1 = (x_1, -x_2), r_3^2 = |X_2 Y|^2 = (y_1 - x_1)^2 + (y_2 + v)^2$, where $X_2 = (x_1, -v)$ and $v \geq x_2, r_4^2 = |X_3 Y|^2 = (y_1 - x_1)^2 + (y_2 + x_2 + s)^2$, where $X_3 = (x_1, -x_2 - s)$ and $s \geq x_2 + y_2, r_5^2 = |X_4 Y_1|^2 = (y_1 - x_1)^2 + s^2$, where $X_4 = (x_1, s)$ and $Y_1 = (y_1, 0), r_6^2 = |XY_1|^2 = (y_1 - x_1)^2 + x_2^2$ and $r_7^2 = |XY_2|^2 = (y_1 - x_1)^2 + (x_2 + s)^2$, where $Y_2 = (y_1, -s)$. Let

$$P = \{(X, Y): |x_1| \leq A, 0 < C_1 \leq x_2 \leq C_2, |y_1| \leq A, 0 < C_1 \leq y_2 \leq C_2\},$$

where A, C_1, C_2 are positive constants and let

$$L_1(X, Y) = E \int_{x_2}^{\infty} \exp(h(v - x_2)) r_3^{2m-2} \ln r_3^2 dv,$$

$$I_3(X, Y) = E \int_0^{\infty} e^{hs} r_4^{2m-2} \ln r_4^2 ds,$$

$$I_2(X, Y) = E \int_{x_2 + y_2}^{\infty} \exp(h(s - x_2 - y_2)) r_5^{2m-2} \ln r_5^2 ds,$$

where $E = h(\pi)^{-1}$.

Now we shall prove

Lemma 1. If $X, Y \in P$, then there exist the integrals $I_i (i = 1, 2, 3)$, which are uniformly convergent in the set P and

$$(3) \quad I_1(X, Y) = I_2(X, Y) = I_3(X, Y).$$

Proof. We shall prove that the integral $I_1(X, Y)$ is uniformly convergent in P . Indeed, we have the inequalities

$$|I_1(X, Y)| \leq E \int_{C_1}^{\infty} e^{hv} e^{-hx_2} r_3^{2m-2} |\ln r_3^2| dv \leq E_1 J_1(X, Y)$$

where

$$J_1(X, Y) = \int_{C_1}^{\infty} e^{hv} r_3^{2m-2} |\ln r_3^2| dv \quad \text{and} \quad E_1 = Ae^{-hC_2}.$$

Let

$$J_1(X, Y) = J_2(X, Y) + J_3(X, Y)$$

where

$$J_2(X, Y) = \int_{C_1}^R e^{hv} r_3^{2m-2} |\ln r_3^2| dv \quad \text{and} \quad J_3(X, Y) = \int_R^{\infty} e^{hv} r_3^{2m-2} |\ln r_3^2| dv.$$

For the arbitrary number $R > C_1$ and $X, Y \in P$ there exists the integral $J_2(X, Y)$. Let ε be an arbitrary positive number. There exists the number $R(P)$ such that $r_3^2 < 4v^2$ and $|\ln r_3^2| < \ln v^2 < 4v^2$ for $v > R > C_1$ and $X, Y \in P$. From the last inequality we get

$$\begin{aligned} |J_3(X, Y)| &\leq \int_R^{\infty} e^{hv} (2v)^{2m-2} 4v^2 dv = 2^{2m} \int_R^{\infty} (e^{Hv})^{-1} v^{2m} dv \\ &\leq 2^m \int_R^{\infty} (2m+2)! v^{2m} (H^{2m+2} v^{2m+2})^{-1} dv = E_2 R^{-1}, \end{aligned}$$

where

$$H = -h \quad \text{and} \quad E_2 = 2^m (2m+2)! (H^{2m+2})^{-1}.$$

If $R > (\varepsilon)^{-1} E_2$, then the integral $J_3(X, Y) < \varepsilon$ for every $X, Y \in P$. Hence the integral $J_1(X, Y)$ is uniformly convergent in the set P .

For the integrals $I_i (i = 2, 3)$ the common majorant is similar to the majorant of the integral $I_1(X, Y)$.

Now we shall prove condition (3). Applying in the integral $I_1(X, Y)$ a change of variable

$$(4) \quad y_2 + v = s$$

we get $I_1(X, Y) = I_2(X, Y)$. Introducing in the integral $I_1(X, Y)$ the new variable

$$(5) \quad v - x_2 = s$$

we obtain $I_1(X, Y) = I_3(X, Y)$.

Let

$$I_{pq}(X, Y) = E \int_0^{\infty} e^{hs} |D_{y_1 y_2} (r_5^{2m-2} \ln r_5^2)| ds$$

and

$$I_{pq}^R(X, Y) = E \int_R^{\infty} e^{hs} |D_{y_1 y_2} (r_5^{2m-2} \ln r_5^2)| ds,$$

where $p, q = 0, 1, \dots, 2m-2$, and $p+q \leq 2m-2$.

From the triangle inequality it follows that there exists a number R such that

$$(6) \quad \frac{1}{4}s^2 \leq r_4^2 \leq 4s^2$$

for every $s > R$ and all $X, Y \in P$.

We shall prove

Lemma 2. The integrals $I_{pq}(X, Y)$ are uniformly convergent in every set P .

Proof. The integrals

$$\int_0^R e^{hs} D_{y_1 y_2} (r_5^{2m-2} \ln r_5^2) ds$$

are convergent for $X, Y \in P$. By (6) the integrals $I_{pq}^R(X, Y)$ have the common majorant

$$(7) \quad C \int_R^\infty e^{hs} s^{2m} ds$$

for a sufficiently great R , C being a constant. Let ε be an arbitrary positive number. Since the integral (7) is convergent, we obtain the inequality $I_{pq}^R(X, Y) < \varepsilon$ for $R > R_0(\varepsilon) > 0$ and all $X, Y \in P$, therefore the thesis of Lemma 2.

2. Now we shall prove.

Theorem 1. The function

$$G(X, Y) = (2\pi)^{-1} r_1^{2m-2} \ln r_1^2 + (2\pi)^{-1} r_2^{2m-2} \ln r_2^2 + I_1(X, Y)$$

is the Green function for problems (1), (2).

Proof. By Lemmas 1, 2 we get

$$\begin{aligned} \Delta_Y^m G(X, Y) &= \Delta_Y^m ((2\pi)^{-1} r_1^{2m-2} \ln r_1^2) + \Delta_Y^m ((2\pi)^{-1} r_2^{2m-2} \ln r_2^2) + \\ &+ E \int_0^\infty e^{hs} \Delta_Y^m (r_4^{2m-2} \ln r_4^2) ds = 0. \end{aligned}$$

Since the functions $r_i^{2m-2} \ln r_i^2$ ($i = 1, 2, 4$) are m -harmonic functions with respect to Y , we shall prove that the function $G(X, Y)$ satisfies the homogeneous boundary conditions (2).

Applying formula (3) and the formula

$$(8) \quad \Delta_Y^i (r_j^{2m-2} \ln r_j) = A_{i,j} r_j^{2m-2i-2} \ln r_j + C_{i,j} r_j^{2m-2i-2} = v_i(r_j),$$

where $i = 0, 1, \dots, m-1$, $j = 1, 2, 4$; $C_{i,j} = C_i$ being constants and $A_{0,j} = 1$, $A_{i,j} = 4^i (i-1) \dots (m-i)^2 = A_i$, $i \geq 1$, we obtain

$$\Delta_Y^i G(X, Y) = (2\pi)^{-1} (v_i(r_1) + v_i(r_2)) + E \int_0^\infty e^{hs} v_i(r_4) ds$$

and using (4), (5) we get

$$\Delta_Y^i G(X, Y) = (2\pi)^{-1} (v_i(r_1) + v_i(r_2)) - E \int_{x_2+y_2}^\infty \exp(h(s-x_2-y_2)) v_i(r_5) ds.$$

Hence

$$\begin{aligned} (D_{y_2} + h) \Delta_Y^i G(X, Y)|_{y_2=0} &= (v_i'(r_1) D_{y_2} r_1 + v_i'(r_2) D_{y_2} r_2 + h(v_i(r_1) + v_i(r_2)))|_{y_2=0} + \\ &+ D_{y_2} (E \int_{x_2+y_2}^\infty \exp(h(s-x_2-y_2)) v_i(r_5) ds)|_{y_2=0} + hE (\int_{x_2+y_2}^\infty e^{h(s-x_2-y_2)} v_i(r_5) ds|_{y_2=0}). \end{aligned}$$

Since

$$(v'_i(r_1)D_{y_2}r_1 + v'_i(r_2)D_{y_2}r_2)|_{y_2=0} + h(v_i(r_1) + v_i(r_2))|_{y_2=0} = Ev_i(r_6),$$

$$h \left(\int_{x_2+y_2}^{\infty} \exp(h(s-x_2-y_2))v_i(r_5)ds \right)|_{y_2=0} = hE \int_{x_2}^{\infty} \exp(h(s-x_2))v_i(r_5)ds$$

and

$$ED_{y_2} \left(\int_{x_2+y_2}^{\infty} \exp(h(s-x_2-y_2))v_i(r_5)ds \right)|_{y_2=0} = -hE \int_{x_2}^{\infty} \exp(h(s-x_2))v_i(r_5)ds - Ev_i(r_6).$$

Hence we have $(D_{y_2} + h)A_Y^i G(X, Y)|_{y_2=0} = 0$.

3. Now we shall solve the problem (1), (2) by means of the Green function $G(X, Y)$. We shall prove that under certain assumptions concerning the functions f_i , the function

$$(9) \quad u(X) = \sum_{i=0}^{m-1} u_i(X)$$

where

$$(9a) \quad u_i(X) = B_i \int_{-\infty}^{+\infty} f_i(s) A_Y^{m-i-1} G(X, y_1, 0) dy_1,$$

and $B_i \cdot A_{i,j} = 1$, is the solution of the problem (1), (2).

In virtue of (8), (9) and (9a) we get

$$u_i(X) = V_i(X) + W_i(X),$$

where

$$V_i(X) = 2B_i \int_{-\infty}^{+\infty} f_i(y_1) K_i(X, y_1, 0) dy_1, \quad W_i(X) = EB_i \int_{-\infty}^{+\infty} f_i(y_1) \left(\int_0^{\infty} e^{hs} L_i(X, y_1, s) ds \right) dy_1$$

and where

$$K_i(X, y_1, 0) = (2\pi)^{-1} (A_i r_6^{2m-2i-2} \ln r_6^2 + C_i r_6^{2m-2i-2}),$$

$$L_i(X, y_1, s) = (\pi)^{-1} (A_i r_7^{2m-2i-2} \ln r_7^2 + C_i r_7^{2m-2i-2}).$$

Let

$$V_{ipq}(X) = B_i \int_{-\infty}^{+\infty} f_i(y_1) D_{x_1^p x_2^q} K_i(X, y_1, 0) dy_1,$$

$$W_{ipq}(X) = B_i \int_{-\infty}^{+\infty} \int_0^{\infty} f_i(y_1) e^{hs} D_{x_1^p x_2^q} L_i(X, y_1, s) ds dy_1,$$

and

$$V_{ipq}^R(X) = B_i \int_{|y_1| > R} |f_i(y_1) D_{x_1^p x_2^q} K_i(X, y_1, 0)| dy_1,$$

$$W_{ipq}^R(X) = B_i \int_{|y_1| > R} \int_R^{\infty} |f_i(y_1) e^{hs} D_{x_1^p x_2^q} L_i(X, y_1, s)| ds dy_1,$$

where $p, q = 0, 1, \dots, 2m, p+q \leq 2m$ and let

$$P_1 = \{(x_1, x_2): |x_1| \leq A, 0 < C_1 \leq x_2 \leq C_2\}.$$

4. In order to prove this proposition we shall need some lemmas.

Lemma 3. If the functions f_i are bounded, continuous and the integral

$$\int_{|y_1| > R} |f_i(y_1)| y_1^{2m-2} \ln |y_1| dy_1$$

is convergent for every $R > 0$, then the integrals $V_{ipq}(X)$ and $W_{ipq}(X)$ are uniformly convergent with respect to X in every set P_1 .

Proof. The integrals

$$\int_{|y_1| < R} f_i(y_1) D_{x_1^p x_2^q} K_i(X, y_1, 0) dy_1 \text{ and } \int_{|y_1| < R} \int_0^R f_i(y_1) e^{hs} D_{x_1^p x_2^q} L_i(X, y_1, s) ds dy_1$$

are convergent for $X \in P_1$. Let ε denote an arbitrary positive number. Applying the inequality (6) for $R > R_0(\varepsilon)$, we get

$$V_{ipq}^R(X) \leq C^1 \int_{|y_1| > R} |f_i(y_1)| y_1^{2m} \ln |y_1| dy_1 < \varepsilon$$

and

$$W_{ipq}(X) \leq C^2 \int_{|y_1| \geq R} |f_i(y_1)| |y_1|^{2m} \ln |y_1| dy_1 < \varepsilon$$

C^1, C^2 being positive constants. Hence we obtain the thesis of the Lemma 3.

By Lemma 3 we get

Lemma 4. If the functions f_i satisfy the assumptions of Lemma 3, then there exist the derivatives $D_{x_1^p x_2^q} V_i(X)$, $D_{x_1^p x_2^q} W_i(X)$ and

$$D_{x_1^p x_2^q} V_i(X) = V_{ipq}(X), \quad D_{x_1^p x_2^q} W_i(X) = W_{ipq}(X).$$

Lemma 5. If the functions f_i satisfy the assumptions of Lemma 3 then the functions $u_i(X)$ are m -harmonic in the half-plane $x_2 > 0$.

Proof. Applying Lemmas 1, 4 and formula (8a) we obtain

$$\Delta^m u_i(X) = B_i \int_{-\infty}^{+\infty} f_i(y_1) \Delta_y^i \Delta_X^m G(X, y_1, 0) dy_1 = 0.$$

Since $\Delta_X^m G(X, y_1, 0) = 0$, thus $u(X)$ as a sum of m -harmonic functions is m -harmonic.

5. Let us now pass to the proof of the boundary conditions (2). Assuming that the functions satisfy the conditions of Lemma 3 and certain other conditions we shall prove that

$$(10) \quad \lim(D_{x_2} - h) \Delta^i u_i(X) = f_i(x_1^0) \text{ as } X \rightarrow (x_1^0, 0+) \quad (i = 0, 1, \dots, m-1)$$

Lemma 6. If the functions f_i satisfy the assumptions of Lemma 3, then the functions u_i satisfy the conditions (10).

Proof. By ([1] p. 263) and Lemma 4 we have

$$(D_{x_2} + h) \Delta^i u_i(X) = (\pi)^{-1} \int_{-\infty}^{+\infty} f_i(y_1) x_2 ((y_1 - x_1)^2 + x_2^2)^{-1} dy_1 \rightarrow f_i(x_1^0).$$

Lemma 7. Let the functions f_i satisfy the assumptions of Lemma 3 and $j \leq i$, then

$$(D_{x_2} + h)\Delta^j u_i(X) = M_i x_2 \int_{-\infty}^{+\infty} f_i(y_1) (r_6^{2m-2i-2+2j} \ln r_6^2 + N_i r_6^{2m-2i-2+2j}) dy_1,$$

M_i, N_i being constants.

Proof. By formula 8 and Lemma 4 we easily obtain the thesis of Lemma 7.

Lemma 8. If the functions f_i satisfy the assumptions of Lemma 3, then

$$\Delta^j u_i(X) = 0 \quad \text{for } j > i \quad \text{and } X \in P_1.$$

Proof. By Lemma 4 and Theorem 1 we get

$$\Delta^j u_i(X) = B_i \int_{-\infty}^{+\infty} f_i(y_1) \Delta_Y^{m-i-1+j} G(X, y_1, 0) dy_1 = 0.$$

Using Lemmas 5, 6, 8 we shall prove the fundamental

Theorem 2. If 1° the functions f_i ($i = 0, 1, \dots, m-1$) are bounded and continuous, 2° the integrals

$$\int_{|y_1| > R} |f_i(y_1) y_1^{2m-2} \ln |y_1|| dy_1$$

are convergent for every $R > 0$, 3° for $i > j$ $\lim (D_{x_2} + h)\Delta^j u_i(X) = 0$ as $X \rightarrow (x_1^0, 0^+)$, then the function $u(X)$ defined by formulae (8), (8a) is the solution of the problem (1), (2).

Proof. By Lemma 5 the function $u(X)$ is m -harmonic in the half-plane $x_2 > 0$. From Lemmas 6, 8 and assumption 3° of Theorem 2 the limit conditions (2) follow. Well by Lemma 6 for $i = j$ we obtain

$$(D_{x_2} + h)\Delta^j u_i(X) \rightarrow f_j(x_1^0, 0^+) \quad \text{as } X \rightarrow (x_1^0, 0^+)$$

and from Lemma 8 for $j > i$ we get

$$(D_{x_2} + h)\Delta^j u_i(X) \rightarrow 0 \quad \text{when } X \rightarrow (x_1^0, 0^+).$$

Remark. If the functions f_i are continuous for $y_1 \in (-\infty, +\infty)$ and equal zero for $y_1 \in R^1 \setminus K$; K being a compact, $K \subset K_1 = \{y_1: |y_1| > R_0 > 0\}$, then the conditions 2° and 3° of Theorem 2 are satisfied.

REFERENCES

- [1] M. Krzyżański, *Partial Differential Equations of Second Order*, vol. I, Warszawa 1971.